Triangulated categories
Bachelor's project

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Abstract

In this paper, we give a short introduction to the theory of triangulated categories. We present the relevant definitions and properties of triangulated categories which we use to investigate the Verdier localisation of triangulated categories. Furthermore, we show that the stable category of a Frobenius category is triangulated. By exhibiting the category of chain complexes as a Frobenius category we may take its stable category, which turns out to coincide with the homotopy category, thus showing that the homotopy category is triangulated. Then, by Verdier localising the homotopy category with respect to the subcategory of acyclic complexes (resp. $\mathcal{X}$-acyclic complexes), we obtain the derived category (resp. $\mathcal{X}$-relative derived category), proving it to be triangulated. Finally, using the theory of triangulated categories, we prove that the (Gorenstein) derived category of an abelian category $\mathcal{A}$ is abelian if and only if $\mathcal{A}$ is semisimple.

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Contents

0 Introduction

The purpose of this paper is to present a short introduction to the theory of triangulated categories. To this end, we knit together pieces of Thorsten Holm and Peter Jørgensens article *Triangulated categories: definitions, properties and examples* (see [HJ10]), Dieter Happels book *Triangulated categories in the representation of finite dimensional algebras* (see [Hap88]) and Amnon Neemans book *Triangulated categories* (see [Nee01]). Most of the proofs presented can be found in at least one of these sources (unless cited otherwise). However we have added detail to- and/or revised many of these proofs. Additionally, we provide an original proof, when necessary. Also note that this paper is not completely self-contained as we omit the proof of a few lemmas, especially in Section 2.

Triangulated categories were first invented by Jean-Louis Verdier in the early 1960's in an effort to axiomatize derived categories. They have since then spread far and wide within mathematics and have become indispensable in many areas of mathematics including topology, algebra, geometry and representation theory.

The success of triangulated categories can partially be attributed to the simple and rich framework which they provide. Indeed, they allow us to talk about important categories (such as the derived category) and their relations in an abstract and axiomatised setting, in much the same manner as abelian categories. Furthermore, they provide a rich theory which allows us to prove many statements which one might formulate within this special setting.

Triangulated categories, as presented in this text, do however face a number of problems, with the non-functoriality of the cone construction being the most problematic. There are a number of alternative definitions of triangulated categories, stemming from stable ∞-categories, dg-categories or stable derivators, which attempt to fix this problem (and succeed in doing so). However, these definitions should not be thought of as a replacement for the traditional definition, but rather as complementary, since they themselves have the problem of being rather difficult to work with, in regards to constructing and working with examples.

In this text we will present the traditional definition of the triangulated category, as well as its related constructions and properties, such as the Verdier localisation. Furthermore we will look at the stable category of a Frobenius category and show it is triangulated. Finally, we will present some important examples of triangulated categories, namely the homotopy category and the derived category. We will also take a quick look at relative derived categories and the special case of the Gorenstein derived category, as they can quite easily be defined using the theory we build up.

0.1 Preliminaries

It is assumed that the reader is accustomed to category theory and general homological algebra. In this short section we state (without proof) a few of the important recurring definitions and propositions used throughout this paper.

Definition 0.1.1 *An* abelian category *is a category $\mathcal{A}$ which satisfies the axioms:*

A1 *$\mathcal{A}$ admits a zero object, finite products and finite coproducts.*

A2 $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ *is an abelian group such that composition of morphisms is bilinear.*

A3 *Kernels and cokernels always exist.*

A4 *For any map $f : X \to Y$, the natural morphism* $\mathrm{coim}(f) \to \mathrm{im}(f)$ *is an isomorphism.*

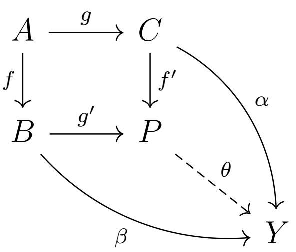
If $\mathcal{A}$ satisfies **A1** *and* **A2**, *then we say that $\mathcal{A}$ is an* additive category.

Remark 0.1.2 In fact, **A2** implies that finite products and finite coproducts coincide. A partial converse is that if $\mathcal{A}$ admits finite coproducts, finite products and they coincide, then $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ can be equipped with a commutative monoid structure by defining $f+g$ as the composition $X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$, where Δ (resp. ∇) is the diagonal (resp. codiagonal).

Example 0.1.3 The category ${}_R\mathbf{Mod}$ of left R-modules over a ring R is abelian. The category of chain complexes $\mathrm{ch}(\mathcal{A})$ (defined in Section 4.1) over an abelian category $\mathcal{A}$ is abelian.

Definition 0.1.4 *An* additive functor $F : \mathcal{C} \to \mathcal{D}$ *is a functor between additive categories $\mathcal{C}$ and $\mathcal{D}$ such that F is an abelian group homomorphism on the* Hom *sets.*

Definition 0.1.5 *In a category $\mathcal{C}$, a* pushout *of two morphisms $f : A \to B$, $g : A \to C$ is an object P together with two morphisms $f' : C \to P$, $g' : B \to P$ with $f'g = g'f$, such that whenever there exists an object Y and two morphisms $\alpha : C \to Y$, $\beta : B \to Y$ such that $f\beta = g\alpha$, then there exists a map $\theta : P \to Y$ such that the following diagram commutes:*



Pushouts are unique up to isomorphism. The dual of a pushout is a *pullback*, however we will only use pushouts throughout this paper.

Proposition 0.1.6 *Let $\mathcal{A}$ be an abelian category. Then a commutative square, as in Definition 0.1.5, is a pushout square if and only if* $\mathrm{coker}(g) \cong \mathrm{coker}(g')$.

Definition 0.1.7 *Let $\mathcal{C}$ be a category and $\sim$ a congruence relation on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (i.e. an equivalence relation such that $f_1 \sim f_2$ and $g_1 \sim g_2$ implies that $g_1 f_1 \sim g_2 f_2$ for all $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$). We define the* quotient category $\mathcal{C}/\sim$ *as having the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}/\sim}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/\sim$.*

Proposition 0.1.8 *If $\mathcal{C}$ is additive, $\sim$ a congruence relation on the Hom sets and if $f_1 \sim f_2$ and $g_1 \sim g_2$ implies that $f_1 + g_1 \sim f_2 + g_2$, then the quotient category $\mathcal{C}/\sim$ is additive. In this case we call $\sim$ an* additive congruence relation.

1 Triangulated categories

We begin by stating the basic definitions, giving intuitions and proving properties of triangulated categories which will be useful in later sections. We closely follow Thorsten Holm and Peter Jørgensens article *Triangulated categories: definitions, properties and examples* ([HJ10]). Most examples of triangulated categories are quite intricate and thus examples will be omitted until Section 4.

1.1 Definition of a triangulated category

Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an additive functor with an inverse Σ^{-1} such that $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = \mathrm{id}_{\mathcal{T}}$, where $\mathrm{id}_{\mathcal{T}}$ is the identity functor of $\mathcal{T}$. We call such a functor Σ an *automorphism* on $\mathcal{T}$, but in the context of triangulated categories we call Σ the *shift functor* on $\mathcal{T}$.

Definition 1.1.1 *A* candidate triangle *in $\mathcal{T}$ is a sequence $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$.*

A (iso-)morphism of candidate triangles is a triple (f, g, h) of (iso-)morphisms such that the following diagram commutes

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

Definition 1.1.2 *A* triangulated category $(\mathcal{T}, \Sigma, \Delta)$ *is an additive category $\mathcal{T}$ equipped with a shift functor Σ and a class Δ of candidate triangles, called* exact triangles *(or simply* triangles *for short), which satisfy the following axioms*

TR0 *Any candidate triangle isomorphic to an exact triangle is an exact triangle.*

TR1 *The candidate triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$ is an exact triangle for any object X in $\mathcal{T}$.*

TR2 *For every $f : X \to Y$ in $\mathcal{T}$ there exists an exact triangle of the form $X \xrightarrow{f} Y \to Z \to \Sigma X$. We call the object Z the cone of f and sometimes denote it by* $\mathrm{cone}(f)$.

TR3 *If the candidate triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle, then the candidate triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ must also be an exact triangle.*

TR4 *Any commutative diagram of the form*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & {\scriptstyle \Sigma f}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

where the rows are exact triangles, can be completed to a morphism of triangles (but not necessarily uniquely!).

TR5 *Given three exact triangles* $X \xrightarrow{u} Y \to Z' \to \Sigma X$, $Y \xrightarrow{v} Z \to X' \to \Sigma Y$, $X \xrightarrow{vu} Z \to Y' \to \Sigma X$ *there exists an exact triangle* $Z' \to Y' \to X' \to \Sigma Z'$ *which makes the following diagram commute*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle v} & & \downarrow & & \| \\
X & \xrightarrow{vu} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
{\scriptstyle u}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & \Sigma Y \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \Sigma Z'
\end{array}$$

We call Δ the *triangulation* of $\mathcal{T}$. In broad strokes, we should think of exact triangles as corresponding to exact sequences in abelian categories. However, in abelian categories, exact sequences are a natural trait of the underlying category, while triangulated categories require the extra data, Σ and Δ. Thus we must note that triangulation is not necessarily unique and an additive category may have several different triangulated structures [HJ10].

Axiom **TR5** may seem a bit arbitrary at first glance, but the axiom is more intuitive than one might initially think. Indeed, given two morphisms $u : X \to Y$, $v : Y \to Z$ in $\mathcal{T}$, we have a morphism $vu : X \to Z$ and thus by **TR2** we have three triangles $X \xrightarrow{u} Y \to Z' \to \Sigma X$, $Y \xrightarrow{v} Z \to X' \to \Sigma Y$, $X \xrightarrow{vu} Z \to Y' \to \Sigma X$. Then **TR4** says that we have two morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{vu} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
{\scriptstyle u}\downarrow & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & \Sigma Y \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Z' & \overset{f}{\dashrightarrow} & Y' & \overset{g}{\dashrightarrow} & X' & \longrightarrow & \Sigma Z'
\end{array}$$

commutes. Subsequently, **TR5** allows us to choose f and g in such a way that the candidate triangle $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \to \Sigma Z'$ becomes an exact triangle. The resulting slogan is that *we may combine **TR2** and **TR4** in a way that is compatible with the exact structure.*

Another way to understand **TR5** is as encoding a version of the third isomorphism theorem for triangulated categories. To elaborate, thinking of triangles as exact sequences would suggest that Z' should be thought of as the quotient of X and Y, i.e. $Y/X \cong Z'$. Likewise, $Z/X \cong Y'$ and, by **TR5**, $Y'/Z' \cong X'$. Then we have $(Z/X)/(Y/X) \cong Y'/Z' \cong X' \cong Z/Y$, which notationally says the same as the third isomorphism theorem of, e.g., groups.

1.2 Triangulated functors and triangulated subcategories

We will now present a few definitions which will aid us in describing relationships between triangulated categories. These definitions will be indispensable for Section 2 and are in general quite central for the study of triangulated categories.

Definition 1.2.1 *A* triangulated functor *between triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ is an additive functor $F : \mathcal{T} \to \mathcal{T}'$ which commutes with the shift functor (i.e. there is a natural isomorphism $\Sigma F \cong F\Sigma$) and maps exact triangles to exact triangles.*

We call a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ a triangle-equivalence *if F is also an equivalence of categories. In that case, $\mathcal{T}$ and $\mathcal{T}'$ are called* triangle-equivalent.

Definition 1.2.2 *An isomorphism-closed, full and additive subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is called a* triangulated subcategory *if $\Sigma\mathcal{C} = \mathcal{C}$ (i.e. the shift functor on $\mathcal{D}$ is also a shift functor on $\mathcal{C}$) and if for any triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{D}$ with $X, Y \in \mathcal{C}$, the object Z is an object of $\mathcal{C}$ as well.*

To demystify this definition, $\mathcal{C}$ is a triangulated subcategory of $\mathcal{D}$ if $\mathcal{C}$ is a triangulated category with the triangulated structure of $\mathcal{D}$. So, $\Sigma\mathcal{C} = \mathcal{C}$ simply implies that $\mathcal{C}$ can inherit the shift functor from $\mathcal{D}$. The last condition of the definition is to ensure that if $f : X \to Y$ is a morphism in $\mathcal{C}$, then the induced triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$ in $\mathcal{D}$ is also a triangle in $\mathcal{C}$. It is easily seen that $\mathcal{C}$ is then a triangulated category.

Definition 1.2.3 *A triangulated subcategory $\mathcal{C}$ of $\mathcal{D}$ is called* thick *if whenever X, Y are objects of $\mathcal{D}$ and $X \oplus Y$ is an object of $\mathcal{C}$, then both X and Y are objects of $\mathcal{C}$ (i.e. $\mathcal{C}$ contains all direct summands of its objects).*

Definition 1.2.4 *Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor. The* kernel *of F, denoted* $\ker(F)$*, is the full subcategory of $\mathcal{D}$ whose objects consist of objects $X \in \mathcal{T}$ such that $FX \cong 0$.*

Lemma 1.2.5 *The kernel of a triangulated functor F is a thick triangulated subcategory.*

Proof. Obviously, $\ker(F)$ is isomorphism-closed, full and additive. We also see that if $X \in \mathcal{C}$, then $\Sigma X \in \mathcal{C}$ since $FX \cong 0$, so $0 \cong \Sigma(FX) \cong F(\Sigma X)$ by the natural isomorphism $\Sigma F \cong F\Sigma$. Furthermore, if $f : X \to Y$ is a morphism in $\mathcal{C}$, then by applying F to $X \xrightarrow{f} Y \to Z \to \Sigma X$ we obtain an exact triangle that is isomorphic to $0 \xrightarrow{0} 0 \to FZ \to 0$, implying that $FZ \cong 0$.

To prove the thickness of $\ker(F)$, we simply note that, by additivity of F, we have that if $X \oplus Y \in \mathcal{C}$, then $0 \cong F(X \oplus Y) = FX \oplus FY$, which implies that $FX \cong 0$ and $FY \cong 0$. □

Definition 1.2.6 *A* (covariant) cohomological functor *$H : \mathcal{T} \to \mathcal{A}$ is an additive functor from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$ such that when $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is an exact triangle, then the sequence*

$$\cdots \to H(\Sigma^i X) \xrightarrow{H(\Sigma^i u)} H(\Sigma^i Y) \xrightarrow{H(\Sigma^i v)} H(\Sigma^i Z) \xrightarrow{H(\Sigma^i w)} H(\Sigma^{i+1} X) \to \dots$$

with $i \in \mathbb{Z}$, is exact.

By duality we also have a notion of a contravariant cohomological functor.

1.3 Properties of triangulated categories

Of course, these categories admit some nice properties of which we present the ones that we deem important and which are generally useful when working with triangulated categories. In our case, we will see them in use in Section 2 and Section 4.

Throughout this subsection we let $\mathcal{T}$ be a triangulated category with shift functor Σ.

Proposition 1.3.1 *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be an exact triangle in $\mathcal{T}$. Then $vu = 0$ and $wv = 0$.*

Proof. On the strength of **TR3** it suffices to show that $vu = 0$. By **TR1** we have a commutative diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X & \xrightarrow{-\Sigma u} & \Sigma Y \\ {\scriptstyle v}\downarrow & & \| & & & & \downarrow{\scriptstyle \Sigma v} \\ Z & = & Z & \xrightarrow{0} & 0 & \xrightarrow{0} & \Sigma Z \end{array}$$

where the top row is an exact triangle by **TR3**. Using **TR4** we may complete the diagram such that $0 = \Sigma(v)(-\Sigma u) = -\Sigma(vu)$, from which we deduce that $vu = 0$, because Σ is additive. □

Proposition 1.3.2 *For any object T of $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(T, -)$ is a cohomological functor.*

Proof. In virtue of **TR3** it is enough to show that, for some $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i X) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(T,\Sigma^i u)} \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i Y) \xrightarrow{\mathrm{Hom}_{\mathcal{T}}(T,\Sigma^i v)} \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i Z)$$

is an exact sequence in **Ab** by showing that $\mathrm{im}(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i u)) = \ker(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i v))$. By Proposition 1.2.1 we have $\Sigma^i(v)\Sigma^i(u) = 0$, so

$$\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i v)\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i u) = \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i(v)\Sigma^i(u)) = \mathrm{Hom}_{\mathcal{T}}(T, 0) = 0$$

showing that $\mathrm{im}(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i u)) \subseteq \ker(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i v))$.

Conversely, let $f \in \mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i Y)$ such that $\Sigma^i(v)f = 0$ (i.e. $f \in \ker(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i v))$. Then, using **TR1** with **TR3**, we obtain a commutative diagram where the rows are triangles

$$\begin{array}{ccccccc} \Sigma^{-i}T & \xrightarrow{0} & 0 & \xrightarrow{0} & \Sigma^{-i+1}T & \xrightarrow{-\mathrm{id}} & \Sigma^{-i+1}T \\ \downarrow{\scriptstyle \Sigma^{-i}f} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle \Sigma^{-i+1}f} \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X & \xrightarrow{-\Sigma u} & \Sigma Y \end{array}$$

This diagram can be completed using **TR4**, so that there exists $h : \Sigma^{-i+1}T \to \Sigma X$ such that $-\Sigma(u)h = -\Sigma^{-i+1}f$, which implies that $f = \Sigma^i(u)\Sigma^{i-1}(h)$. Hence $f \in \mathrm{im}(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i u))$, and thus $\mathrm{im}(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i u)) \supseteq \ker(\mathrm{Hom}_{\mathcal{T}}(T, \Sigma^i v))$ as desired. □

In fact, by duality one can also show that $\mathrm{Hom}_{\mathcal{T}}(-, T)$ is a contravariant cohomological functor.

Proposition 1.3.3 (Triangulated 5-lemma) *Let (f, g, h) be a morphism of triangles. If any two among f, g and h are isomorphisms, then so is the third.*

Proof. By **TR3** we only need to prove the case where f and g are isomorphisms.

We are given $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$. Thus, we have the following diagram

$$\begin{array}{ccccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Y \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & {\scriptstyle \Sigma f}\big\downarrow & & {\scriptstyle \Sigma g}\big\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' & \longrightarrow & \Sigma Y' \end{array}$$

on which we may apply $\mathrm{Hom}_{\mathcal{T}}(Z', -)$, yielding the diagram

$$\begin{array}{ccccccccc} \mathrm{Hom}(Z',X) & \longrightarrow & \mathrm{Hom}(Z',Y) & \longrightarrow & \mathrm{Hom}(Z',Z) & \longrightarrow & \mathrm{Hom}(Z',\Sigma X) & \longrightarrow & \mathrm{Hom}(Z',\Sigma Y) \\ \big\downarrow{\scriptstyle \mathrm{Hom}(Z',f)} & & {\scriptstyle \mathrm{Hom}(Z',g)}\big\downarrow & & {\scriptstyle \mathrm{Hom}(Z',h)}\big\downarrow & & {\scriptstyle \mathrm{Hom}(Z',\Sigma f)}\big\downarrow & & {\scriptstyle \mathrm{Hom}(Z',\Sigma g)}\big\downarrow \\ \mathrm{Hom}(Z',X') & \longrightarrow & \mathrm{Hom}(Z',Y') & \longrightarrow & \mathrm{Hom}(Z',Z') & \longrightarrow & \mathrm{Hom}(Z',\Sigma X') & \longrightarrow & \mathrm{Hom}(Z',\Sigma Y') \end{array}$$

of which the rows are exact by Proposition 1.3.2. Since f and g are isomorphisms, Σf and Σg are also isomorphisms. Thus, $\mathrm{Hom}(Z', f)$, $\mathrm{Hom}(Z', g)$, $\mathrm{Hom}(Z', \Sigma f)$, $\mathrm{Hom}(Z', \Sigma g)$ are all isomorphisms. By applying the 5-lemma for abelian categories we deduce that $\mathrm{Hom}(Z', h)$ is an isomorphism, which implies that there exists $h^{-1} \in \mathrm{Hom}(Z', Z)$ such that $hh^{-1} = \mathrm{id}_{Z'}$. Using the same argument but with the functor $\mathrm{Hom}_{\mathcal{T}}(-, Z')$ we produce a left inverse h'^{-1} such that $h'^{-1}h = \mathrm{id}_Z$. Finally, since $h'^{-1} = h'^{-1}(hh^{-1}) = (h'^{-1}h)h^{-1} = h^{-1}$, we see that h is an isomorphism with inverse h^{-1}. □

Corollary 1.3.4 *For any morphism $f : X \to Y$, $\mathrm{cone}(f)$ is unique up to (non-canonical) isomorphism. Furthermore, f is an isomorphism if and only if $\mathrm{cone}(f) \cong 0$.*

Proof. The first statement follows from applying Proposition 1.3.3 to the following diagram

$$\begin{array}{ccccccc} X & = & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \| & & \| & & \big\downarrow & & \| \\ X & = & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \end{array}$$

The second statement is shown by using **TR3** and **TR4** to obtain the diagram on the left and **TR4** to obtain the diagram on the right:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \| & & \big\downarrow & & \| & & \| \\ X & = & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \end{array} \qquad \begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \| & & \big\downarrow{\scriptstyle f^{-1}} & & \big\downarrow & & \| \\ X & = & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \end{array}$$

By proposition 1.3.3 the constructed morphisms $X \to Y$ and $Z \to 0$ are isomorphims. Thus the $\Leftarrow$ implication is a result of the left hand diagram and the $\Rightarrow$ implication is a result of the right hand diagram. □

The converse of **TR3** is usually a part of the definition of a triangulated category, but, as the following lemma will show, it is not necessary as it can be deduced from the other axioms.

Lemma 1.3.5 *If $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is an exact triangle then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is also an exact triangle.*

Proof. Using **TR2** on $u : X \to Y$ we have an exact triangle $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$. Repeated use of **TR3** then yields the exact triangles $\Sigma X \xrightarrow{-\Sigma u} \Sigma Y \xrightarrow{-\Sigma v} \Sigma Z \xrightarrow{-\Sigma w} \Sigma^2 X$ and $\Sigma X \xrightarrow{-\Sigma u} \Sigma Y \xrightarrow{-\Sigma v'} \Sigma Z' \xrightarrow{-\Sigma w'} \Sigma^2 X$. Hence we may consider the following commutative diagram wherein f is a consequence of **TR4**:

$$\begin{array}{ccccccc}
\Sigma X & \xrightarrow{\Sigma u} & \Sigma Y & \xrightarrow{\Sigma v} & \Sigma Z & \xrightarrow{\Sigma w} & \Sigma^2 X \\
\| & & \| & & \downarrow{\scriptstyle f} & & \| \\
\Sigma X & \xrightarrow{\Sigma u} & \Sigma Y & \xrightarrow{\Sigma v'} & \Sigma Z' & \xrightarrow{\Sigma w'} & \Sigma^2 X
\end{array}$$

By Proposition 1.3.3 we know f to be an isomorphism, which implies that $\Sigma^{-1} f : Z \to Z'$ is an isomorphism. Hence the following is an isomorphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle \Sigma^{-1} f} & & \| \\
X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X
\end{array}$$

where the bottom row is an exact triangle by assumption. By **TR0** we may thus conclude that the top row is also an exact triangle. □

We finish off this section with a rather nice characterization of the overlap of abelian and triangulated categories.

Definition 1.3.6 *We call an abelian category $\mathcal{A}$* semisimple *if every short exact sequence in $\mathcal{A}$ splits. Equivalently, $\mathcal{A}$ is semisimple if and only if $\mathcal{A}$ is abelian and every morphism $f : X \to Y$ has a pseudo-inverse $g : Y \to X$, i.e. $fgf = f$ and $gfg = g$.*

Lemma 1.3.7 *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be an exact triangle with $w = 0$. Then u is a split monomorphism and v is a split epimorphism. We say that the triangle* splits.

Proof. To show u is a split monomorphism, we will construct $u' : Y \to X$ such that $u'u = \mathrm{id}_X$. Since $w = 0$, the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\| & & & & {\scriptstyle 0}\downarrow & & \| \\
X & = & X & \longrightarrow & 0 & \xrightarrow{0} & \Sigma X
\end{array}$$

commutes, and by **TR3** and **TR4** it may be completed with a morphism $u' : Y \to X$ which has the desired properties.

The proof that v is a split epimorphism follows analogously. □

This next theorem will prove particularly useful in Section 4, when we prove that the derived category of $\mathcal{A}$ is abelian if and only if $\mathcal{A}$ is semi simple.

Theorem 1.3.8 *Let $\mathcal{T}$ be a triangulated and abelian category. Then $\mathcal{T}$ is semisimple.*

Proof. Given any exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, we wish to show that f is a split monomorphism, since then the sequence will split by the splitting lemma.

By **TR2** we have $X \xrightarrow{f} Y \xrightarrow{u} V \xrightarrow{v} \Sigma X$, which **TR3** turns into $V \xrightarrow{v} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma u} \Sigma V$. Applying Σ^{-1} then yields $\Sigma^{-1}V \xrightarrow{\Sigma^{-1}v} X \xrightarrow{f} Y \xrightarrow{u} V$, from which we observe that $f\Sigma^{-1}(v) = 0$, by Proposition 1.3.1. But f is a monomorphism, so $\Sigma^{-1}v = 0$, which implies that $v = 0$. Hence, by Lemma 1.3.7, the exact triangle $X \xrightarrow{f} Y \xrightarrow{u} V \xrightarrow{v} \Sigma X$ splits and f is a split monomorphism, as desired. □

In fact, the convese statement holds true as well. Let $\Sigma = \mathrm{id}_{\mathcal{A}}$ be the shift functor and define the candidate triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ to be an exact triangle if and only if it is exact at all three objects. Then one can prove that this structure is a triangulated structure on any semisimple category $\mathcal{A}$ (see [Fri14]).

As an example, since the category of finite-dimensional vector spaces $\mathbf{FinVect}_k$ over a field k is semisimple, it follows that $\mathbf{FinVect}_k$ is triangulated.

2 The Verdier localisation of triangulated categories

One weakness of triangulated categories is their lack of constructions of new triangulated categories from old ones. However, Verdier's localisation theorem is one of the few theorems that allows us to do just that: Construct new triangulated categories from a given triangulated category $\mathcal{D}$ and any triangulated subcategory $\mathcal{C} \subset \mathcal{D}$.

As the name suggests, this theorem is due to Jean-Louis Verdier and says the following:

Verdier's Localisation Theorem *Let $\mathcal{C}$ be a triangulated subcategory of a triangulated category $\mathcal{D}$. Then there exists a triangulated category $\mathcal{D}/\mathcal{C}$, as well as a triangulated functor $F_{univ} : \mathcal{D} \to \mathcal{D}/\mathcal{C}$, such that $\mathcal{C}$ is a subcategory of $\ker(F_{univ})$, and F_{univ} is universal with the following property:*

If $F : \mathcal{D} \to \mathcal{T}$ is a triangulated functor such that $\mathcal{C}$ is a subcategory of $\ker(F)$, then F factors uniquely through $\mathcal{D}/\mathcal{C}$ as $\mathcal{D} \xrightarrow{F_{univ}} \mathcal{D}/\mathcal{C} \to \mathcal{T}$.

Our exposition will mainly follow Amnon Neemans *Triangulated categories* ([Nee01]), with slight variation at times and a bit of inspiration from Fabian Lenzen ([Len15]).

We will not be giving the (quite technical) proof that $\mathcal{D}/\mathcal{C}$ is triangulated. However we will be showing how the category $\mathcal{D}/\mathcal{C}$ is constructed and partly prove that it is an additive category, as well as proving some lemmas here and there. Furthermore, we will construct the functor F_{univ} and prove its universality. All proofs omitted can be found in [Nee01].

We will ignore any set-theoretic complications that could arise.

2.1 Construction and categoriality of $\mathcal{D}/\mathcal{C}$

The objects of $\mathcal{D}/\mathcal{C}$ are simply the same objects as in $\mathcal{D}$. The morphisms however are a bit more intricate; we start by defining the following collection:

Definition 2.1.1 Let $\mathcal{C}$ *be a triangulated subcategory of $\mathcal{D}$. We define* $\mathrm{Mor}_{\mathcal{C}}$ *as the collection of morphisms $f : X \to Y$ in $\mathcal{D}$ for which* $\mathrm{cone}(f)$ *is an object of $\mathcal{C}$.*

This collection is well-defined since $\mathrm{cone}(f)$ always exists and is unique up to isomorphism.

Definition 2.1.2 *In a triangulated category, a commutative square*

$$\begin{array}{ccc} P & \xrightarrow{f} & A \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ B & \xrightarrow{f'} & Q \end{array}$$

is called a homotopy Cartesian *if there exists a triangle $P \xrightarrow{\binom{g}{-f}} B \oplus A \xrightarrow{(f',g')} Q \to \Sigma P$. We call P the* homotopy pullback *of f' and g', and Q the* homotopy pushout *of f and g. We denote the homotopy pullback by $A \times_Q^h B$.*

Remark 2.1.3 By **TR2**, homotopy pullbacks and pushouts always exist, and, by Corollary 1.2.4, homotopy pullbacks and pushouts are unique up to (non-canonical) isomorphism.

Lemma 2.1.4 *The collection* $\mathrm{Mor}_{\mathcal{C}}$ *has the following properties:*

1. $\mathrm{Mor}_{\mathcal{C}}$ *contains all isomorphisms in $\mathcal{D}$.*

2. *If any two of the morphisms f, g, gf lie in* $\mathrm{Mor}_{\mathcal{C}}$*, then so does the third. In particular, we may compose morphisms in* $\mathrm{Mor}_{\mathcal{C}}$*.*
3. *All homotopy pushouts and pullbacks of morphisms in* $\mathrm{Mor}_{\mathcal{C}}$ *are contained in* $\mathrm{Mor}_{\mathcal{C}}$*.*

Proof. 1. Note that $\mathcal{C}$ is additive, so 0 is an object of $\mathcal{C}$. By Corollary 1.3.4 we have that f is an isomorphism if and only if $\mathrm{cone}(f) \cong 0$, and since triangulated subcategories are closed under isomorphisms, it follows that $\mathrm{cone}(f) \in \mathcal{C}$. Hence $f \in \mathrm{Mor}_{\mathcal{C}}$.

2. Using **TR5** we have the following diagram in $\mathcal{D}$:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\| & & \downarrow & & \downarrow{\scriptstyle f'} & & \| \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma Y \\
\downarrow & & \| & & \downarrow{\scriptstyle g'} & & \downarrow \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & \Sigma Z \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Z' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & X' & \longrightarrow & \Sigma Z'
\end{array}$$

where the bottom row is a triangle. We need only show that if two of Z', Y', X' lie in $\mathcal{C}$, then so does the third. But this is true by **TR3** and the condition on triangulated subcategories that if $Z' \to Y' \to X' \to \Sigma Z'$ is in $\mathcal{D}$ with Z', Y' in $\mathcal{C}$, then X' is in $\mathcal{C}$.

3. By **TR3** and uniqueness of cones, the result follows. □

We will later see (in Lemma 2.2.2) that any triangulated functor whose kernel contains $\mathcal{C}$ (in particular F_{univ}) maps the morphisms in $\mathrm{Mor}_{\mathcal{C}}$ to isomorphisms. This justifies our use of the term 'localisation' as opposed to the term 'quotient'.[1]

Definition 2.1.5 *For two objects X, Y of $\mathcal{D}$, we let $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$ denote the collection of diagrams of the form*

$$\begin{array}{ccccc}
 & & W & & \\
 & \swarrow^{f} & & \searrow & \\
X & & & & Y
\end{array}$$

where $f \in \mathrm{Mor}_{\mathcal{C}}$. We call such a diagram a roof.

With these morphism sets we actually obtain a new category, as this next lemma will show.

Definition/Lemma 2.1.6 *We define composition $(X \leftarrow W \to Y) \circ (Y \leftarrow W' \to Z)$ of roofs in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$ as given by the diagram*

$$\begin{array}{ccccccccc}
 & & W & \longleftarrow & W \times^h_Y W' & \longrightarrow & W' & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
X & & & & Y & & & & Z
\end{array}$$

[1] Recall, the term 'localisation' usually refers to the addition of inverses of certain morphisms such that they become isomorphisms, while the term 'quotient' usually refers to the identification of morphisms through some congruence relation $\sim$ on the Hom sets.

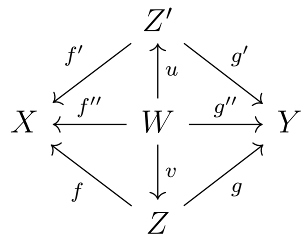
which can be shortened to $(X \leftarrow W \times^h_Y W' \to Z)$.

This composition is associative and has the identity $\widehat{\mathrm{id}}_X := (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X)$.

By Remark 2.1.3, composition of roofs is unique up to isomorphism and thus well-defined.

Proof. Omitted. See [Nee01], Lemma 2.1.19 (pp. 80-81). □

Definition/Lemma 2.1.7 *Let* $\oplus$ *be the relation on* $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$ *defined by* $(X \leftarrow Z \to Y) \oplus (X \leftarrow Z' \to Y)$ *if and only if there exists a roof* $(X \leftarrow W \to Y)$ *and two morphisms* $u : W \to Z'$ *and* $v : W \to Z$ *such that the diagram*



commutes. Then $\oplus$ *is a congruence relation. Furthermore, the morphisms* u *and* v *are contained in* $\mathrm{Mor}_{\mathcal{C}}$*. We denote the equivalence class of* $(X \xleftarrow{f} W \xrightarrow{g} Y)$ *by* gf^{-1}.

Proof. Omitted. See [Nee01], Lemma 2.1.14 and Lemma 2.1.18 (pp. 76-77 and pp. 80). □

The notation gf^{-1} is quite natural since $f \in \mathrm{Mor}_{\mathcal{C}}$ will become invertible in $\mathcal{D}/\mathcal{C}$, and so we should think of gf^{-1} as a composition of morphisms. Furthermore, the existence of a diagram like the one in Definition/Lemma 2.1.7 'implies' that

$$gf^{-1} = guu^{-1}f^{-1} = g''f''^{-1} = g'vv^{-1}f'^{-1} = g'f'^{-1}.$$

So it would be natural to identify $(X \leftarrow Z \to Y)$ with $(X \leftarrow Z' \to Y)$.

Indeed, this is how we define our morphisms in $\mathcal{D}/\mathcal{C}$:

Definition 2.1.8 *As previously stated, the objects of* $\mathcal{D}/\mathcal{C}$ *are simply the objects of* $\mathcal{D}$*. The morphism sets of* $\mathcal{D}/\mathcal{C}$ *are defined as* $\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X, Y) := \widehat{\mathrm{Hom}}_{\mathcal{D}}(X, Y)/\oplus$.

The previous lemmas show that $\mathcal{D}/\mathcal{C}$ is indeed a category, in fact, a quotient category.

2.2 Universality of F_{univ} and Properties of $\mathcal{D}/\mathcal{C}$

Since $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}$, we define the functor $F_{univ} : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ as the following:

$$X \mapsto X$$
$$(X \xrightarrow{f} Y) \mapsto (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y).$$

Lemma 2.2.1 *Let* $f, g : X \to Y$ *be morphisms in* $\mathcal{D}$*. Then* $F_{univ}(f) = F_{univ}(g)$ *if and only if* $f - g : X \to Y$ *factors over a* $Z \in \mathcal{C}$ *as* $X \to Z \to Y$.

Proof. Omitted. See [Nee01], Lemma 2.1.26 (pp. 84-85). □

Thus $F_{univ}(\mathrm{id}_Z) = 0$ for $Z \in \mathcal{C}$, which means that $\ker(F_{univ})$ contains $\mathcal{C}$, as desired.

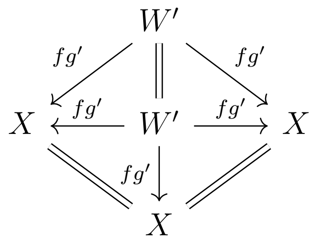
Lemma 2.2.2 *Any triangulated functor* $F : \mathcal{D} \to \mathcal{T}$ *whose kernel contains* $\mathcal{C}$ *maps any morphism in* $\mathrm{Mor}_{\mathcal{C}}$ *to an isomorphism. In particular,* F_{univ} *will have this property.*

Proof. Let $f \in \mathrm{Mor}_{\mathcal{C}}$. We apply F to a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$ where $Z \in \mathcal{C}$. This yields the triangle $FX \xrightarrow{Ff} FY \to FZ \to \Sigma FX$ which is isomorphic to $FX \xrightarrow{Ff} FY \to 0 \to \Sigma FX$. By Corollary 1.3.4, Ff is an isomorphism. □

Lemma 2.2.3 *Let gf^{-1} be a morphism in $\mathcal{D}/\mathcal{C}$. The following statements hold:*

1. *If $g \in \mathrm{Mor}_{\mathcal{C}}$, then gf^{-1} is invertible with the inverse $(gf^{-1})^{-1} = fg^{-1}$.*
2. *We can write gf^{-1} as $gf^{-1} = F_{univ}(g)F_{univ}(f)^{-1}$, where $f \in \mathrm{Mor}_{\mathcal{C}}$ and g in $\mathcal{D}$.*
3. *The morphism $F_{univ}f$ is an isomorphism if and only if $f \in \mathrm{Mor}_{C}$.*

Proof. 1. Notice that $(X \xleftarrow{f} W \xrightarrow{g} Y) \circ (Y \xleftarrow{g} W \xrightarrow{f} X) = (X \xleftarrow{fg'} W' \xrightarrow{fg'} X)$, and by Lemma 2.1.4 (statements 2 and 3), we have that $fg' \in \mathrm{Mor}_{\mathcal{C}}$, so $(X \xleftarrow{fg'} W' \xrightarrow{fg'} X) \oplus (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X)$ by the following:

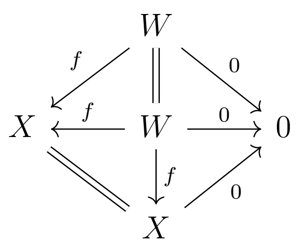


Analogously, $(X \xleftarrow{f} W \xrightarrow{g} Y) \circ (Y \xleftarrow{g} W \xrightarrow{f} X) = (Y \xleftarrow{\mathrm{id}_Y} Y \xrightarrow{\mathrm{id}_Y} Y)$.

2. This is clear, since $(X \xleftarrow{f} W \xrightarrow{\mathrm{id}_W} W) \circ (W \xleftarrow{\mathrm{id}_W} W \xrightarrow{g} Y) = (X \xleftarrow{f} W \xrightarrow{g} Y)$.
3. The $\Leftarrow$ implication is just Lemma 2.2.2. Assume that $F_{univ}f = f\mathrm{id}_X^{-1}$ is an isomorphis. Then its inverse is $\mathrm{id}_X f^{-1} = F_{univ}(\mathrm{id}_X)F_{univ}(f)^{-1}$, which implies that $f \in \mathrm{Mor}_{\mathcal{C}}$. □

Lemma 2.2.4 *The category $\mathcal{D}/\mathcal{C}$ is additive, and the functor F_{univ} is additive.*

Proof. A qualified guess for the zero object of $\mathcal{D}/\mathcal{C}$ would be the zero object 0 of $\mathcal{D}$. Indeed, the following diagram shows that $0\ \mathrm{id}_X^{-1} \oplus 0f^{-1}$, i.e. that 0 is a terminal object:



Analogously, 0 is also an initial object.

For biproducts the obvious guess is $X \oplus_{\mathcal{D}/\mathcal{C}} Y = X \oplus_{\mathcal{D}} Y$, which indeed does satisfy the universal property for product and coproduct in $\mathcal{D}/\mathcal{C}$. Furthermore, F_{univ} respects these coproducts. See [Nee01], Lemma 2.1.29 (pp. 87-89) for the long and quite technical proof of these two facts. We also find that $F_{univ}(0) \cong 0$, which means that F_{univ} is an additive functor.

It remains only to equip the Hom sets of $\mathcal{D}/\mathcal{C}$ with an abelian group structure. By Remark 0.1.2, $\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X,Y)$ already has a commutative monoid structure given by the composition $f+g : X \to X \oplus X \xrightarrow{f\oplus g} Y \oplus Y \to Y$, which we will show is an abelian groups structure:

By Lemma 2.2.3 (2) we may write any morphism in $\mathcal{D}/\mathcal{C}$ as $F_{univ}(g)F_{univ}(f)^{-1}$ which has the group inverse $F_{univ}(-g)F_{univ}(f)^{-1}$. Indeed, since F_{univ} is an additive functor we have that

$$F_{univ}(g)F_{univ}(f)^{-1} + F_{univ}(-g)F_{univ}(f)^{-1} = F_{univ}(g-g)F_{univ}(f)^{-1} = F_{univ}(0)F_{univ}(f)^{-1} = 0.$$

Thus $\mathrm{Hom}_{\mathcal{D}/\mathcal{C}}(X,Y)$ has an abelian group structure. □

Theorem 2.2.5 *On the category $\mathcal{D}/\mathcal{C}$ we define the shift functor $\Sigma : \mathcal{D}/\mathcal{C} \to \mathcal{D}/\mathcal{C}$ as:*

$$X \mapsto \Sigma_{\mathcal{D}} X$$
$$(X \xleftarrow{f} W \xrightarrow{g} Y) \mapsto (\Sigma_{\mathcal{D}} X \xleftarrow{\Sigma_{\mathcal{D}} f} \Sigma_{\mathcal{D}} W \xrightarrow{\Sigma_{\mathcal{D}} g} \Sigma_{\mathcal{D}} Y)$$

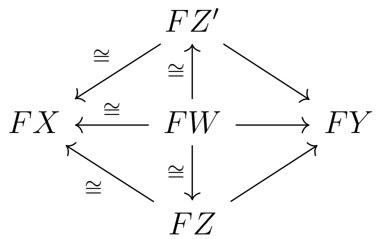
where $\Sigma_{\mathcal{D}}$ is the shift functor on $\mathcal{D}$.

Let the exact triangles of $\mathcal{D}/\mathcal{C}$ consist of the candidate triangles in $\mathcal{D}/\mathcal{C}$ which are isomorphic to the image of an exact triangle in $\mathcal{D}$ under F_{univ}. The shift functor together with the triangles exhibit $\mathcal{D}/\mathcal{C}$ as a triangulated category.

Proof. Omitted. See [Nee01], *Proof of Theorem 2.1.8* (pp.97-99). □

Proposition 2.2.6 *The functor $F_{univ} : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is triangulated and satisfies the following universal property: Every triangulated functor $F : \mathcal{D} \to \mathcal{T}$ sending* $\mathrm{Mor}_{\mathcal{C}}$ *to isomorphisms (in particular, functors whose kernel contain $\mathcal{C}$) factors as $\mathcal{D} \xrightarrow{F_{univ}} \mathcal{D}/\mathcal{C} \to \mathcal{T}$.*

Proof. By definition of F_{univ} and Lemma 2.2.4, F_{univ} is triangulated. Let $F : \mathcal{D} \to \mathcal{T}$ be a functor which maps all morphisms in $\mathrm{Mor}_{\mathcal{C}}$ to isomorphisms. Then we may extend F to any roof in $\widehat{\mathrm{Hom}}_{\mathcal{D}}(X,Y)$ in the obvious way. By applying F is two equivalent roofs we get



We see that F sends two roofs, which are equivalent modulo $\oplus$, to the same morphism. □

Proposition 2.2.7 *The functor F_{univ} maps a morphism $X \to Y$ to an isomorphism if and only if for any triangle $X \to Y \to Z \to \Sigma X$ there exists $Z' \in \mathcal{D}$ such that $Z \oplus Z' \in \mathcal{C}$.*

Proof. Omitted. See [Nee01], Proposition 2.1.35 (pp. 92-94). □

By letting $X = 0$ we get that $Z \cong Y$ and $F_{univ}(Y) \cong 0$ if and only if there exists $Z' \in \mathcal{D}$ so that $Y \oplus Z' \in \mathcal{C}$. Thus, $\ker(F_{univ})$ is the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{C}$.

3 The stable category of a Frobenius category

In this section we will present the stable category of a Frobenius category, which bears many fruitful examples of triangulated categories, many of which are especially useful in algebra. The goal of the section is to construct the stable Frobenius category, whereafter we equip it with a shift functor and a triangulation which, together, exhibit it as a triangulated category. In the ensuing section we will look at the important example that is the homotopy category of chain complexes which turns out to be a stable Frobenius category.

The exposition of this section primarily follows *Triangulated categories in the representation of finite dimensional algebras* by Dieter Happel ([Hap88]) but also takes inspiration and a few proofs from *Triangulated categories: definitions, properties and examples*" by Thorsten Holm and Peter Jørgensen ([HJ10]). However, as Happel is quite sparing with the details, we present our own proofs for Lemma 3.1.7 and Proposition 3.2.4 as well as a lot of added detail to most of the proofs, most notably in the proof of Theorem 3.3.2.

3.1 Definition of the stable category of a Frobenius category

Definition 3.1.1 *Let $\mathcal{A}$ be a category with a notion of short exact sequences. We say that an object X is an* extension *of Q by K if there exists a short exact sequence of the form*

$$0 \to K \to X \to Q \to 0.$$

A subcategory, which also has a notion of short exact sequences, $\mathcal{B}$ of $\mathcal{A}$, is said to be closed under extension *if whenever Q and K are in $\mathcal{B}$, then X is isomorphic to an object in $\mathcal{B}$.*

Definition 3.1.2 *Let $\mathcal{B}$ be an additive, full and extension-closed subcategory of some abelian category $\mathcal{A}$. Let $\mathcal{E}$ be the class of all short exact sequences in $\mathcal{A}$ with entries in $\mathcal{B}$.*

We call the pair $(\mathcal{B}, \mathcal{E})$ an exact category*.*

Exact categories were first introduced by David Quillen, and they allow us to have a notion of exactness in a category that is not necessarily abelian. Quillen worked out a long list of axioms for exact categories, but it was then later shown that any exact category $\mathcal{B}$ can be fully embedded into an abelian category $\mathcal{A}$ such that $\mathcal{B}$ is extension-closed and all short exact sequences in $\mathcal{B}$ become actual short exact sequences in the ambient abelian category $\mathcal{A}$.

Example 3.1.3 *Any abelian category $\mathcal{A}$ together with the class $\mathcal{E}$ of all short exact sequences in $\mathcal{A}$ is trivially an exact category as it can be embedded into itself, in the manner of Definition 3.1.2. This is the 'finest' exact category in the sense that we cannot have more exact sequences originating from $\mathcal{A}$.*

Example 3.1.4 *Any abelian category $\mathcal{A}$ with the class $\mathcal{E}$ of split exact sequences in $\mathcal{A}$ is also an exact category. This is trivially seen using Quillens original axioms but one can also exhibit an ambient category. Indeed, by the Yoneda lemma, we have a fully faithful embedding $\mathcal{A} \to \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ab})$, sending $X \mapsto \mathrm{Hom}(-, X)$ which is full, extension-closed and satisfies the condition that a sequence in $\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ab})$ is exact if and only if it is split exact in $\mathcal{A}$. This is the 'coarsest' exact category in the sense that we cannot have fewer exact sequences originating from $\mathcal{A}$.*

Let $(\mathcal{B}, \mathcal{E})$ be an exact category. If $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is in $\mathcal{E}$, we call u a *proper monomorphism* (or simply *monic*) and v a *proper epimorphism* (or simply *epic*).

We call an object P of $\mathcal{B}$ an *$\mathcal{E}$-projective* object if whenever $v : Y \to Z$ is epic and $f : P \to Z$ is any morphism in $\mathcal{B}$, then there exists a morphism $g : P \to Y$ such that $f = vg$.

Dually, we call an object I of $\mathcal{B}$ an *$\mathcal{E}$-injective* object if whenever $u : X \to Y$ is monic and $f : X \to I$ is any morphism in $\mathcal{B}$, then there exists a morphism $g : Y \to I$ such that $f = gu$.

We say that the exact category $(\mathcal{B}, \mathcal{E})$ has enough $\mathcal{E}$-projectives (resp. enough $\mathcal{E}$-injectives) if for any object X in $\mathcal{B}$ there exists a proper epimorphism (resp. proper monomorphism) $v : P \to X$ (resp. $u : X \to I$).

These $\mathcal{E}$-projective and $\mathcal{E}$-injective behave almost exactly like regular projective and injective objects in an abelian category, e.g. the splitting lemma holds in exact categories (the proof is exactly the same).

Definition 3.1.5 *A* Frobenius category *is an exact category with enough projectives, enough injectives and where the classes of projectives and injectives coincide.*

Example 3.1.6 (Frobenius algebras) Let K be a field. We call a finite-dimensional K-algebra Λ a *Frobenius algebra* if there exists a linear form $\pi : \Lambda \to K$ whose kernel does not contain any non-zero left ideal of Λ. Then the category of finite-dimensional Λ-modules is a Frobenius category [Far05].

Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category. For two morphisms $f, g : X \to Y$ in $\mathcal{B}$ we define the relation $\sim_i$ by saying $f \sim_i g$ if and only if $f - g$ factors over an injective object.

Lemma 3.1.7 *The relation $\sim_i$ is an additive congruence relation.*

Proof. Reflexivity and symmetry are clear. To see transitivity we note that if $f - g = \beta_1\alpha_1 : X \to I_1 \to Y$ factors over I_1 and $g - h = \beta_2\alpha_2 : X \to I_2 \to Y$ factors over I_2, then $f - h = f - g + g - h = \beta_1\alpha_1 + \beta_2\alpha_2$ factors over $I_1 \oplus I_2$ which is again injective. To see this, let $\iota_i : I_i \to I_1 \oplus I_2$, $i \in \{1, 2\}$, be the canonical inclusion and $\pi_i : I_1 \oplus I_2 \to I_i$ be the the canonical projection. Then

$$\begin{aligned}(\beta_1\pi_1 + \beta_2\pi_2)(\iota_1\alpha_1 + \iota_2\alpha_2) &= \beta_1\pi_1\iota_1\alpha_1 + \beta_1\pi_1\iota_2\alpha_2 + \beta_2\pi_2\iota_1\alpha_1 + \beta_2\pi_2\iota_2\alpha_2 \\ &= \beta_1\alpha_1 + 0 + 0 + \beta_2\alpha_2 = f - h.\end{aligned}$$

Next we need to show that if $f_1 \sim_i f_2$ and $g_1 \sim_i g_2$, then $g_1f_1 \sim_i g_2f_2$ for all $f_1, f_2 \in \mathrm{Hom}(X, Y)$ and $g_1, g_2 \in \mathrm{Hom}(Y, Z)$. Since $g_1(f_1 - f_2) + (g_1 - g_2)f_2$ factors over an injective object (using the same argument as for transitivity), the claim is shown as follows:

$$g_1(f_1 - f_2) + (g_1 - g_2)f_2 = g_1f_1 - g_1f_2 + g_1f_2 - g_2f_2 = g_1f_1 - g_2f_2.$$

Lastly we need to show that $f_1 + g_1 \sim_i f_2 + g_2$. But this follows from the fact that $(f_1 + g_1) - (f_2 + g_2) = (f_1 - f_2) + (g_1 - g_2)$ factors through an injective object, since $f_1 - f_2$ and $g_1 - g_2$ does (again, using the same argument as for transitivity). □

Definition 3.1.8 *Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category. The* stable Frobenius category $\underline{\mathcal{B}}$ *of $(\mathcal{B}, \mathcal{E})$ is the category which has the same objects as $\mathcal{B}$ and $\mathrm{Hom}_{\underline{\mathcal{B}}}(X, Y) := \mathrm{Hom}_{\mathcal{B}}(X, Y)/\sim_i$. We denote the residue class of a morphism $u : X \to Y$ as $\underline{u}$. Furthermore, $\underline{\mathcal{B}}$ is additive, by Proposition 0.1.8.*

Example 3.1.9 (Dual numbers) Let K be a field. The K-algebra $\Lambda := K[x]/(x^2)$ is a Frobenius algebra with $\pi(a + bx) = b$. Then the category of finite-dimensional Λ-modules is Frobenius and the only indecomposable objects are Λ and K, of which only Λ is injective. Thus, in the stable category, we have $\Lambda \cong 0$, since $\mathrm{id}_\Lambda \sim_i 0$. Hence, $\mathrm{Hom}(\Lambda, \Lambda) = \mathrm{Hom}(\Lambda, K) = \mathrm{Hom}(K, \Lambda) = 0$. However, $\mathrm{Hom}(K, K) \neq 0$, since $\mathrm{id}_K \not\sim_i 0$.

3.2 Construction of the shift functor

Since a Frobenius category $\mathcal{B}$ has enough injectives, enough projectives and they coincide, any object X of $\mathcal{B}$ admits a short exact sequence $0 \to X \xrightarrow{\iota_X} I_X \xrightarrow{\pi_X} Z_X \to 0$ (we will omit the 0's whenever we deem them redundant), where I_X is injective (and projective).

Lemma 3.2.1 *Given a morphism $u : X \to Y$, there exist morphisms $\overline{u} : I_X \to I_Y$ and $u_Z : Z_X \to Z_Y$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota_X} & I_X & \xrightarrow{\pi_X} & Z_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \overline{u}} & & \downarrow{\scriptstyle u_Z} & & \\
0 & \longrightarrow & Y & \xrightarrow{\iota_Y} & I_Y & \xrightarrow{\pi_Y} & Z_Y & \longrightarrow & 0
\end{array}$$

Proof. Since I_Y is injective, ι_X is monic and $\iota_Y u : X \to I_Y$ is a morphism there exists a morphism $\overline{u} : I_X \to I_Y$ making the left square commute. In $\mathcal{A}$, the map ι_X exhibits Z_X as a cokernel, and since $\pi_X \iota_X = 0$ and $\pi_Y \overline{u} \iota_X = \pi_Y \iota_Y u = 0$, it follows from the universal property of cokernels that there exists a unique morphism $u_Z : Z_X \to Z_Y$ such that $\pi_Y \overline{u} = u_Z \pi_X$. $\mathcal{B}$ is a full subcategory of $\mathcal{A}$, so u_Z will also be a morphism in $\mathcal{B}$. □

Lemma 3.2.2 *The morphism u_Z is independent of the chosen morphism $\overline{u}$ and independent of the chosen representative of u in $\underline{\mathcal{B}}$. In particular, u_Z is unique in $\underline{\mathcal{B}}$.*

Proof. Let $\overline{u}$ and $\overline{u'}$ be two liftings of u inducing u_Z and u'_Z respectively. Observe that

$$0 = \iota_Y u - \iota_Y u = \overline{u}\iota_X - \overline{u'}\iota_X = (\overline{u} - \overline{u'})\iota_X.$$

Since Z_X is the cokernel of ι_X and $(\overline{u} - \overline{u'})\iota_X = 0$, it follows from the universal property of cokernels that there exists a morphism $\sigma : Z_X \to I_Y$ such that $\sigma\pi_X = \overline{u} - \overline{u'}$. By commutivity we get

$$\pi_Y \sigma \pi_X = \pi_Y(\overline{u} - \overline{u'}) = \pi_Y \overline{u} - \pi_Y \overline{u'} = u_Z \pi_X - u'_Z \pi_X = (u_Z - u'_Z)\pi_X.$$

Since π_X is epic we may cancel π_X from the right yielding $\pi_Y \sigma = u_Z - u'_Z$, which means $u_Z - u'_Z$ factors through an injective object. Hence $u_Z = u'_Z$ in $\underline{\mathcal{B}}$, which proves the first claim.

For the second claim, we wish to show that if u factors through an injective object, then so does u_Z. Assume $u = u''u' : X \to I \to Y$ factors through I, then we have the following:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u'"] \arrow[dd, bend right, "u"'] & I_X \arrow[r] \arrow[d, "\overline{u'}"] & Z_X \arrow[d, "u'_Z"'] \arrow[dd, bend left, "u_Z"] \\ I \arrow[r] \arrow[d, "u''"] & I_I \arrow[r] \arrow[d, "\overline{u''}"] & Z_I \arrow[d, "u''_Z"'] \\ Y \arrow[r] & I_Y \arrow[r] & Z_Y \end{tikzcd}$$

where each row is exact and $\overline{u'}$, $\overline{u''}$, u'_Z, u''_Z are results of Lemma 3.2.1 and make the diagram commute. We note that since I is injective the diagram splits, so the injective object I_I is isomorphic to $I \oplus Z_I$, which we claim implies that Z_I must also be injective. Indeed, if we have a morphism $A \to Z_I$, a monomorphism $A \to A'$ and if $Z_I \oplus I$ is injective, then there exists $Z_I \oplus I \to A'$, and by composing with the canonical inclusion $Z_I \to Z_I \oplus I$ the claim follows. Hence, u_Z factors through the injective object Z_I. $\square$

Lemma 3.2.3 *Let $(\mathcal{B}, \mathcal{E})$ be an exact category and let $0 \to X \xrightarrow{\iota} I \xrightarrow{\pi} Z \to 0$ and $0 \to X \xrightarrow{\iota'} I' \xrightarrow{\pi'} Z' \to 0$ be in $\mathcal{E}$, where I, I' are $\mathcal{E}$-injectives. Then Z and Z' are isomorphic in $\underline{\mathcal{B}}$.*

Proof. By Lemma 3.2.1 we have morphisms $f : I \to I'$, $f' : I' \to I$ and $g : Z \to Z'$, $g' : Z' \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, equal] & I \arrow[r, "\pi"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & X \arrow[r, "\iota'"] \arrow[d, equal] & I' \arrow[r, "\pi'"] \arrow[d, "f'"] & Z' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & X \arrow[r, "\iota"] & I \arrow[r, "\pi"] & Z \arrow[r] & 0 \end{tikzcd}$$

By commutivity of the left hand squares we obtain the equality

$$0 = \iota - \iota = f'\iota' - \iota = f'f\iota - \iota = (f'f - \mathrm{id}_I)\iota.$$

Since Z is the cokernel of ι, and $(f'f - \mathrm{id}_I)\iota = 0$, it follows from the universal property of cokernels that there exists a morphism $\sigma : Z \to I$ such that $\sigma\pi = f'f - \mathrm{id}_I$. By commutivity of the right hand squares we obtain the equality

$$\pi\sigma\pi = \pi(f'f - \mathrm{id}_I) = \pi f'f - \pi = g'g\pi - \pi = (g'g - \mathrm{id}_Z)\pi.$$

Since π is epic we may cancel π from the right, yielding $\pi\sigma = g'g - \mathrm{id}_Z$. Thus we observe that the morphism $g'g - \mathrm{id}_Z = \pi\sigma : Z \to I \to Z$ factors through an injective object and is therefore the zero morphism in $\underline{\mathcal{B}}$. Hence we conclude that $g'g = \mathrm{id}_Z$.

Analogously we find that $gg' = \mathrm{id}_{Z'}$ in $\underline{\mathcal{B}}$. Thus Z is isomorphic to Z' in $\underline{\mathcal{B}}$. $\square$

Let $0 \to X \to I \to Z \to 0$ be in $\mathcal{E}$. From this point forward we will assume that there is a bijection

$$\gamma_X : \{X' \mid X' \cong X, X' \in \underline{\mathcal{B}}\} \to \{Z' \mid Z' \cong Z, Z' \in \underline{\mathcal{B}}\},$$

i.e. there is a bijection γ_X between the set of objects isomorphic to X and the set of objects isomorphic to Z. Lemma 2.2.3 shows that this assumption does not depend on the chosen sequence, $0 \to X \to I \to Z \to 0$.

Under this assumption the shift functor Σ, defined below, will be an automorphism. If we do not assume the existence of the bijection γ_X, the shift functor will still be an autoequivalence[2] and any two choices of shift functors will be isomorphic. For the proof of this fact, we refer the reader to [HJ10], Lemma 8.8.

Lastly, it may seem like this assumption will exclude a lot of examples of stable Frobenius categories, but it turns out that most applications and examples do in fact satisfy this assumption.

Definition/Proposition 3.2.4 *Let $\Sigma : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ be the shift functor defined as follows: For any object X of $\underline{\mathcal{B}}$, let $\Sigma X = \gamma_X(X)$ and for a morphism $\underline{u} : X \to Y$ in $\underline{\mathcal{B}}$, let $\Sigma \underline{u} = \underline{u_Z}$. Then Σ is an additive automorphism of $\underline{\mathcal{B}}$.*

Proof. If $u = \mathrm{id}_X$ we may choose $\overline{u} = \mathrm{id}_I$ which will induce $u_Z = \mathrm{id}_{\Sigma X}$. Likewise, if $u = gf : X \to Y \to Z$ we may choose $u_Z = g_Z f_Z : \Sigma X \to \Sigma Y \to \Sigma Z$, which shows that Σ is a functor. Furthermore, given $u, v : X \to Y$ we may take $(u+v)_Z = u_Z + v_Z$, hence $\Sigma(u+v) = u_Z + v_Z = \Sigma(u) + \Sigma(v)$, showing additivity.

We define the inverse functor $\Sigma^{-1} : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ as follows: for an object X of $\underline{\mathcal{B}}$ we let $\Sigma^{-1}X = \gamma_X^{-1}(X)$, and by Lemma 3.2.1 and 3.2.2 the morphism $\underline{u_Z} : \gamma_X(X) \to \gamma_Y(Y)$ exists and is uniquely determined by the morphism $\underline{u} : X \to Y$, hence $\Sigma^{-1}\underline{u_Z} = \underline{u}$ is well-defined. By construction we have $\Sigma\Sigma^{-1} = \Sigma^{-1}\Sigma = \mathrm{id}_{\underline{\mathcal{B}}}$. □

From now on we will write Σu instead of u_Z, even though Σ technically is not defined on morphisms in $\mathcal{B}$.

3.3 Triangulation of the stable Frobenius category

With the shift functor constructed we can move on to defining the class exact triangles Δ of the stable Frobenius category $\underline{\mathcal{B}}$. We will prove they exhibit $\underline{\mathcal{B}}$ as a triangulated category.

Let $0 \to X \to I_X \to \Sigma X \to 0$ be in $\mathcal{E}$ and $u : X \to Y$ a morphism in $\mathcal{B}$. By Proposition 0.1.6 we then have the following diagram in the ambient abelian category $\mathcal{A}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\iota_X} & I_X & \xrightarrow{\pi_X} & \Sigma X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \bar{u}} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & \Sigma X & \longrightarrow & 0 \end{array}$$

where C_u is the pushout of ι_X and u, in $\mathcal{A}$. Since $0 \to Y \to C_u \to \Sigma X \to 0$ is exact (because $\Sigma X \cong \mathrm{coker}(v) \cong \mathrm{coker}(\iota_X)$) and $\mathcal{B}$ is closed under extension we obtain that C_u is also an object of $\mathcal{B}$. This yields a sequence $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} \Sigma X$ in $\underline{\mathcal{B}}$.

[2] Σ being an autoequivalence is sufficient for the definition of triangulated categories in some literatures.

Definition 3.3.1 *We call the sequence* $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$ *in* $\underline{\mathcal{B}}$ *a* standard triangle. *We say that a candidate triangle* $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X$ *in* $\underline{\mathcal{B}}$ *is an exact triangle if it is isomorphic to a standard triangle. We denote the class of all exact triangles in* $\underline{\mathcal{B}}$ *by* Δ*.*

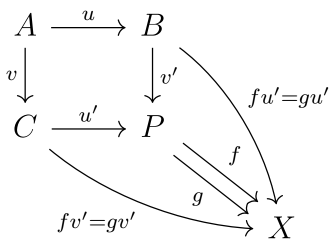
Thus we can finally state and prove the main theorem of this section.

Theorem 3.3.2 *The category* $\underline{\mathcal{B}}$ *equipped with* Σ *and* Δ *is a triangulated category.*

Before we get into the proof we will need a small category-theoretic lemma.

Lemma 3.3.3 *Given a pushout of* $u : A \to B$ *and* $v : A \to C$ *and two morphisms* f *and* g *from the pushout to some object* X*, then* $f = g$ *if and only if* $fu' = gu'$ *and* $fv' = gv'$*, where* u' *and* v' *are the induced maps.*

Proof. The result follows simply from the following diagram, since the map $P \to X$ which makes the diagram commute is unique.



□

Proof of 3.3.2. We check each axiom, **TR0** through **TR5**.
TR0 Δ is closed under isomorphism by definition.

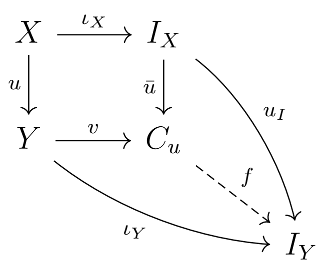
TR1 Since id_I factors through an injective object, namely I itself $\mathrm{id}_I = \mathrm{id}_I \mathrm{id}_I : I \to I \to I$, we must have $\mathrm{id}_I = 0$ in $\underline{\mathcal{B}}$, which implies that $I \cong 0$ in $\underline{\mathcal{B}}$. Then **TR1** follows from the following diagram:

$$\begin{array}{ccccccc} X & = & X & \longrightarrow & I & \longrightarrow & \Sigma X \\ \| & & \| & & \downarrow \cong & & \| \\ X & = & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \end{array}$$

since the upper row is a standard triangle.

TR2 This follows directly from the construction of the standard triangles.

TR3 Given the standard triangle $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$ in $\mathcal{B}$ and letting $Y \to I_Y \to \Sigma Y$ be in $\mathcal{E}$, we may define a morphism $f : C_u \to I_Y$ by the pushout property, as follows:



We note that $\iota_Y = fv$ and $wv = 0$, which yields the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & \Sigma X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_Y} & & \downarrow{\scriptstyle \binom{f}{w}} & & \Vert & & \\
0 & \longrightarrow & I_Y & \xrightarrow{(1,0)} & I_Y \oplus \Sigma X & \xrightarrow{\binom{0}{1}} & \Sigma X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle (\pi_Y, -\Sigma u)} & & & & \\
 & & \Sigma Y & = & \Sigma Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

We note that the first two rows are exact, and since the cokernel of v and $(1,0)$ are equal we know, by Proposition 0.1.6, that the upper left square is a pushout square, and thus the middle column is exact. It follows from the definition of the standard triangles that the image of $Y \xrightarrow{v} C_u \xrightarrow{\binom{f}{w}} I_Y \oplus \Sigma X \xrightarrow{(\iota_Y, -\Sigma u)} \Sigma Y$ in $\underline{\mathcal{B}}$, which is exactly $Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$, is a standard triangle, as desired.

TR4 We first consider the standard triangles. Given two standard triangles $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} C_{u'} \xrightarrow{w'} \Sigma X'$ in $\underline{\mathcal{B}}$, as well as two morphisms $f : X \to X'$ and $g : Y \to Y'$ such that $\underline{u'f} = \underline{gu}$. I.e. we have the following commutative diagram:

$$
\begin{array}{ccccc}
X & \xrightarrow{\iota_X} & I_X & \xrightarrow{\pi_X} & \Sigma X \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \overline{u}} & & \Vert \\
Y & \xrightarrow{v} & C_u & \xrightarrow{w} & \Sigma X \\
 & & & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{\iota_{X'}} & I_{X'} & \xrightarrow{\pi_{X'}} & \Sigma X' \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle \overline{u'}} & & \Vert \\
Y' & \xrightarrow{v'} & C_{u'} & \xrightarrow{w'} & \Sigma X'
\end{array}
\qquad (f : X \to X',\ g : Y \to Y')
$$

Since $\underline{u'f} - \underline{gu} = 0$ we know that $u'f - gu$ factors through an injective object I, and, together with the universal property of injective objects, we obtain a morphism $a : I_X \to I$ such that the following diagram commutes:

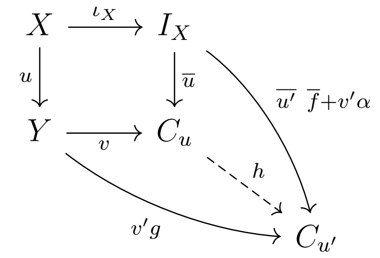
$$
\begin{array}{ccccc}
 & & X & \xrightarrow{u'f - gu} & Y' \\
 & \swarrow{\scriptstyle \iota_X} & & \searrow & \nearrow{\scriptstyle a'} \\
I_X & & \xrightarrow{a} & & I
\end{array}
$$

Letting $\alpha := a'a$ we get a morphism $\alpha : I_X \to Y'$ such that $gu = u'f + \alpha\iota_X$.

By Lemma 3.2.1 we have maps $\overline{f} : I_X \to I_{X'}$ and $\Sigma f : \Sigma X \to \Sigma X'$ such that $\overline{f}\iota_X = \iota_{X'} f$ and $\Sigma(f)\pi_X = \pi_{X'}\overline{f}$. For the two morphisms $v'g : Y \to C_{u'}$ and $\overline{u'}\ \overline{f} + v'\alpha : I_X \to C_{u'}$ we have that

$$v'gu = v'(u'f + \alpha\iota_X) = v'u'f + v'\alpha\iota_X = \overline{u'}\iota_{X'}f + v'\alpha\iota_X = \overline{u'}\ \overline{f}\iota_X + v'\alpha\iota_X = (\overline{u'}\ \overline{f} + v'\alpha)\iota_X.$$

Hence we may define $h : C_u \to C_{u'}$ by the universal property of pushout squares



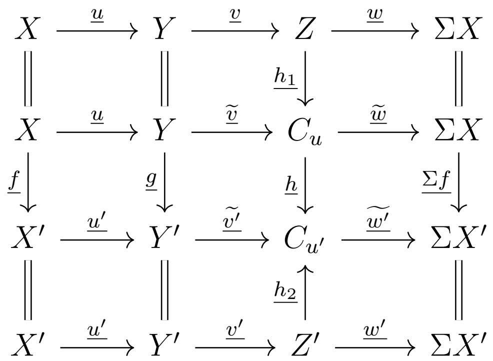
for which $hv = v'g$ and $h\overline{u} = \overline{u'}\ \overline{f} + v'\alpha$.

It remains to show that $w'h = \Sigma(f)w$. To this end it is enough to show that $w'hv = \Sigma(f)wv$ and $w'h\overline{u} = \Sigma(f)w\overline{u}$, by Lemma 3.3.3. Regarding the first equality we observe that $\Sigma(f)wv = 0$ and $w'hv = w'v'g = 0$. The second equality is seen as follows (note: $w'v'\alpha = 0$):

$$\Sigma(f)w\overline{u} = \Sigma(f)\pi_X = \pi_{X'}\overline{f} = w'\overline{u'}\ \overline{f} = w'\overline{u'}\ \overline{f} + w'v'\alpha = w'(\overline{u'}\ \overline{f} + v'\alpha) = w'h\overline{u}.$$

Hence we conclude that $(\underline{f}, \underline{g}, \underline{h})$ is a morphism of triangles.

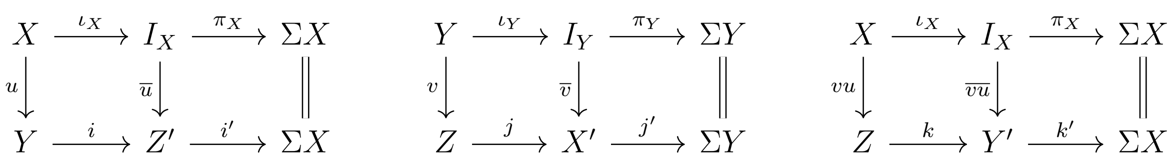
We now turn to the general case. Let $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{w}} \Sigma X$ and $X' \xrightarrow{\underline{u'}} Y' \xrightarrow{\underline{v'}} Z' \xrightarrow{\underline{w'}} \Sigma X'$ be exact triangles in $\underline{\mathcal{B}}$ and let $\underline{f} : X \to X'$ and $\underline{g} : Y \to Y'$ be morphisms such that $\underline{u'f} = \underline{gu}$ in $\underline{\mathcal{B}}$. Since we have isomorphisms of standard triangles we have the following commutative diagram:



where $\underline{h_1}$ and $\underline{h_2}$ are isomorphisms. Then it is clear that $(\underline{f}, \underline{g}, \underline{h_2}^{-1}\underline{h}\underline{h_1})$ is a morphism of triangles. Indeed, $\underline{h_2}^{-1}\underline{h}\underline{h_1}\underline{v} = \underline{h_2}^{-1}\underline{h}\underline{\widetilde{v}} = \underline{h_2}^{-1}\underline{\widetilde{v'}}\underline{g} = \underline{v'}\underline{g}$ and $\underline{w'}\underline{h_2}^{-1}\underline{h}\underline{h_1} = \underline{\widetilde{w'}}\underline{h}\underline{h_1} = \underline{\Sigma f}\underline{\widetilde{w}}\underline{h_1} = \underline{\Sigma f}\underline{w}$.

TR5 It is enough to consider only the standard triangles.

We are given three standard triangles in $\underline{\mathcal{B}}$. I.e. we have the following three diagrams



and we wish to construct $f : Z' \to Y'$ and $g : Y' \to X'$ such that the diagram in axiom **TR5** commutes in $\underline{\mathcal{B}}$, and such that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \to \Sigma Z'$ is a standard triangle.

From the given data we can construct the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & \Sigma X & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle \iota_{Z'}} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & Y & \xrightarrow{\iota_{Z'} i} & I_{Z'} & \xrightarrow{\pi'_Y} & M & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Sigma Z' & = & \Sigma Z' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where M is the pushout of i' and $\iota_{Z'}$. Since $\mathcal{B}$ is closed under extension we may consider the diagram above in $\mathcal{B}$. By Lemma 3.2.3 and by definition of the standard triangles we may consider the right hand diagram below instead of the left hand diagram below

$$\begin{array}{ccccc}
Y & \xrightarrow{\iota_Y} & I_Y & \xrightarrow{\pi_Y} & \Sigma Y\\
{\scriptstyle v}\downarrow & & {\scriptstyle \overline{v}}\downarrow & & \|\\
Z & \xrightarrow{j} & X' & \xrightarrow{j'} & \Sigma Y
\end{array}
\qquad\qquad
\begin{array}{ccccc}
Y & \xrightarrow{\iota_{Z'} i} & I_{Z'} & \xrightarrow{\pi'_Y} & M\\
{\scriptstyle v}\downarrow & & {\scriptstyle \overline{v'}}\downarrow & & \downarrow{\scriptstyle \phi}\\
Z & \xrightarrow{j} & X' & \xrightarrow{j'} & \Sigma Y
\end{array}$$

where ϕ is an isomorphism. Since Σu is unique in $\underline{\mathcal{B}}$ we may assume $\phi p = \Sigma u$. We also note that by the following diagram we have $\Sigma(i)\phi\pi'_Y = \pi_{Z'}$

$$\begin{array}{ccccccc}
Y & \xrightarrow{\iota_{Z'} i} & I_{Z'} & \xrightarrow{\pi'_Y} & M & \xrightarrow{\phi} & \Sigma Y\\
{\scriptstyle i}\downarrow & & \| & & \downarrow & \swarrow{\scriptstyle \Sigma i} & \\
Z' & \xrightarrow{\iota'_Z} & I_{Z'} & \xrightarrow{\pi_{Z'}} & \Sigma Z' & &
\end{array}$$

Since $\overline{vu}\iota_X = kvu$ we may define $f : Z' \to Y'$ using the pushout property as shown on the left hand of the following diagrams. Likewise, since $jvu = \overline{v'}\iota_{Z'}iu = \overline{v'}\iota_{Z'}\overline{u}\iota_X$, we may define $g : Y' \to X'$ using the pushout property as shown on the right hand of the following diagrams:

$$\begin{array}{ccccc}
X & \xrightarrow{\iota_X} & I_X & & \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \overline{u}} & \searrow{\scriptstyle \overline{vu}} & \\
Y & \xrightarrow{i} & Z' & & \\
 & \searrow{\scriptstyle kv} & & \overset{f}{\dashrightarrow} & Y'
\end{array}
\qquad\qquad
\begin{array}{ccccc}
X & \xrightarrow{\iota_X} & I_X & & \\
{\scriptstyle vu}\downarrow & & \downarrow{\scriptstyle \overline{vu}} & \searrow{\scriptstyle \overline{v'}\iota_{Z'}\overline{u}} & \\
Z & \xrightarrow{k} & Y' & & \\
 & \searrow{\scriptstyle j} & & \overset{g}{\dashrightarrow} & X'
\end{array}$$

We now wish to show that the following diagram commutes and that the bottom row is a standard triangle:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & \Sigma X \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & \Sigma X \\
\downarrow{\scriptstyle u} & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow{v} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & \Sigma Y \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} & & \| & & \downarrow{\scriptstyle \Sigma i} \\
Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{\Sigma(i)j'} & \Sigma Z'
\end{array}$$

Ignoring the squares that commute trivially, we see that the top middle square commutes by the left hand pushout square above. The center square commutes by the right hand pushout square above. The bottom left square and the bottom middle square are symmetric to the upper middle square and the center square respectively.

Thus it remains to show that the top right square and the middle right square commute, i.e. that $k'f = i$ and $j'g = \Sigma(u)k'$.

For the first equality, we only need to show that $k'fi = i'i$ and $k'f\overline{u} = i'\overline{u}$, by Lemma 3.3.3 with Z' as the pushout. Indeed, we get $k'fi = k'kv = 0 = i'i$ and $k'f\overline{u} = k'\overline{vu} = \pi_X = i'\overline{u}$.

For the second equality, we again only need show that $j'gk = \Sigma(u)k'k$ and $j'g\overline{vu} = \Sigma(u)k'\overline{vu}$, by Lemma 3.3.3 with Y' as the pushout. Indeed, we get $j'gk = j'j = 0 = \Sigma(u)k'k$ and $j'g\overline{vu} = j'\overline{v'}\iota_{Z'}\overline{u} = \phi\pi'_Y\iota_{Z'}\overline{u} = \phi pi'\overline{u} = \phi p\pi_X = \Sigma(u)\pi_X = \Sigma(u)k'\overline{vu}$.

The very last fact to show is that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{\Sigma(i)j'} \Sigma Z'$ is a standard triangle. In the diagram below, the rectangle is a pushout (recall $j = gk$) and the left hand square is a pushout. Therefore it follows from the pasting law of pushouts that the right hand square is a pushout as well.

$$\begin{array}{ccccc}
Y & \xrightarrow{i} & Z' & \xrightarrow{\iota_{Z'}} & I_{Z'} \\
{\scriptstyle v}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \overline{v}}\downarrow \\
Z & \xrightarrow{k} & Y' & \xrightarrow{g} & X'
\end{array}$$

Since $\Sigma(i)j'\overline{v'} = \Sigma(i)\phi\pi'_Y = \pi_{Z'}$, we thus have the following diagram where the left hand square is a pushout:

$$\begin{array}{ccccc}
Z' & \xrightarrow{\iota_{Z'}} & I_{Z'} & \xrightarrow{\pi_{Z'}} & \Sigma Z' \\
{\scriptstyle f}\downarrow & & {\scriptstyle \overline{v'}}\downarrow & & \| \\
Y' & \xrightarrow{g} & X' & \xrightarrow{\Sigma(i)j'} & \Sigma Z'
\end{array}$$

which exhibits $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{\Sigma(i)j'} \Sigma Z'$ as a standard triangle and thus finishing this god forsaken proof. □

Thus, any exact category with coinciding projectives and injectives induces a triangulated category. These categories (i.e. triangulated categories which are triangle-equivalent to a stable Frobenius category) are called *algebraic* and are used a lot in algebra and representation theory.

4 Examples of triangulated categories

In this section we look the homotopy category and the derived category as examples of triangulated categories. To finish off we will take a quick look at the relative derived categories and the special case of the Gorenstein derived category. All these examples stem from the category of chain complexes, but not every triangulated category is 'of this form'. An important example, not 'of this form', that we omit is the stable module category (and in particular the stable module category over a Frobenius algebra) and can be found in [Hap88].

Subsection 4.2 follows the exposition of Happel, but this section is otherwise mostly our own craft. Especially (because Happel leaves a lot of work to the reader) we give our own proof of Proposition 4.2.2 and Lemma 4.2.3. Furthermore, all lemmas, propositions and theorems, except Theorem 4.3.4, of Subsection 4.3 are our own with help from Henrik Holm and from [Di+14].

4.1 Recalling the definitions

Let $\mathcal{A}$ be an abelian category.

Definition 4.1.1 *A* chain complex*, $X_\bullet$ in $\mathcal{A}$ is a sequence in $\mathcal{A}$*

$$\dots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} \dots$$

such that $d_i d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We call the maps d_i the differentials *of $X_\bullet$.*

A (iso-)morphism of chain complexes *$f_\bullet : X_\bullet \to Y_\bullet$ is a family of (iso-)morphisms $f_i : X_i \to Y_i$, such that the diagram*

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & X_2 & \xrightarrow{d_2^X} & X_1 & \xrightarrow{d_1^X} & X_0 & \xrightarrow{d_0^X} & X_{-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\ \dots & \longrightarrow & Y_2 & \xrightarrow{d_2^Y} & Y_1 & \xrightarrow{d_1^Y} & Y_0 & \xrightarrow{d_0^Y} & Y_{-1} & \longrightarrow & \dots \end{array}$$

commutes.

The category whose objects are chain complexes in $\mathcal{A}$ and whose morphisms are morphisms of chain complexes is called the category of chain complexes and is denoted $\mathrm{ch}(\mathcal{A})$*.*

The full subcategory of left bounded (resp. right bounded, resp. bounded) chain complexes (i.e. the objects are chain complexes such that $X_i = 0$ for all $i > j$ (resp. $i < j$, resp. $|i| < j$) for some $j \in \mathbb{Z}$) is denoted $\mathrm{ch}^+(\mathcal{A})$ (resp. $\mathrm{ch}^-(\mathcal{A})$, resp. $\mathrm{ch}^b(\mathcal{A})$). It is well known that these categories of chain complexes are abelian categories (See, [Wei94] Theorem 1.2.3).

We also remind the reader of the existence of the additive functors H_i defined by $H_i(X_\bullet) = \mathrm{coker}(\mathrm{im}(d_{i+1}) \to \ker(d_i))$, called the homology functors. We say $X_\bullet$ is *acyclic* if $H_i(X_\bullet) \cong 0$ for all $i \in \mathbb{Z}$.

Two morphisms of chain complexes $f_\bullet$ and $g_\bullet$ from $X_\bullet$ to $Y_\bullet$ are said to be *chain homotopic* if there exists a sequence of morphisms $h_i : X_i \to Y_{i+1}$ such that $f_i - g_i = d_{i+1}^Y h_i + h_{i-1} d_i^X$. We say that $f_\bullet$ is *null-homotopic* if it is homotopic to the zero morphism.

Proposition/Definition 4.1.2 *The relation '$f_\bullet$ is chain homotopic to $g_\bullet$', denoted $f_\bullet \sim_h g_\bullet$, is an additive congruence relation.*

We define the homotopy category $K(\mathcal{A})$ *as the quotient category* $\mathrm{ch}(\mathcal{A})/\sim_h$. *In particular, $K(\mathcal{A})$ is an additive category.*

Proof. Omitted. See [HJ10], Proposition 1.7. □

For the sake of completion we will give a rather vague definition of the derived category.

We call a morphism $X_\bullet \to Y_\bullet$ of chain complexes a *quasi-isomorphism* if the induced morphism of homology $H_i(X_\bullet) \to H_i(Y_\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

"Definition" 4.1.3 *The* derived category $D(\mathcal{A})$ *is the category* $\mathrm{ch}(\mathcal{A})$ *localized with respect to quasi-isomorphisms, i.e. quasi-isomorphisms become actual isomorphisms.*

Later in this section we will rigidify this definition by using Verdiers localisation theorem.

4.2 Triangulation of the homotopy category

We now proceed to constructing the class of exact sequences in $\mathrm{ch}(\mathcal{A})$ (as well as $\mathrm{ch}^+(\mathcal{A})$, $\mathrm{ch}^-(\mathcal{A})$, $\mathrm{ch}^b(\mathcal{A})$), which will exhibit it (them) as a Frobenius category, whereafter we show that the corresponding stable category coincides with the homotopy category $K(\mathcal{A})$. From this point forward we simply let $\mathrm{ch}(\mathcal{A})$ represent any of the four chain complex categories, $\mathrm{ch}(\mathcal{A})$, $\mathrm{ch}^+(\mathcal{A})$, $\mathrm{ch}^-(\mathcal{A})$, $\mathrm{ch}^b(\mathcal{A})$, since everything unfolds in the exact same manner for any one of the four.

Let $\mathcal{E}$ be the class of sequences $0 \to X_\bullet \xrightarrow{f_\bullet} Y_\bullet \xrightarrow{g_\bullet} Z_\bullet \to 0$ in $\mathrm{ch}(\mathcal{A})$ for which $0 \to X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \to 0$ is split exact for all $i \in \mathbb{Z}$. For an object X in $\mathcal{A}$ and some $i \in \mathbb{Z}$ we define $I_i(X)_\bullet$ as the complex with X in the i'th and $i+1$'th degree of the complex and 0 otherwise:

$$\cdots \longrightarrow \underset{i+2}{0} \longrightarrow \underset{i+1}{X} = \underset{i}{X} \longrightarrow \underset{i-1}{0} \longrightarrow \underset{i-2}{0} \longrightarrow \cdots$$

Likewise, we define $P_i(X)_\bullet$ as the complex with X in the i'th and $i-1$'th position of the complex and 0 otherwise:

$$\cdots \longrightarrow \underset{i+2}{0} \longrightarrow \underset{i+1}{0} \longrightarrow \underset{i}{X} = \underset{i-1}{X} \longrightarrow \underset{i-2}{0} \longrightarrow \cdots$$

It is then easily seen that $I_i(X)_\bullet$ is an $\mathcal{E}$-injective object and $P_i(X)_\bullet$ is an $\mathcal{E}$-projective object. We will later see that any $\mathcal{E}$-injective (resp. $\mathcal{E}$-projective) is direct summand of an injective object of the form $\bigoplus_{i\in\mathbb{Z}} I_i(X_i)_\bullet$ (resp. $\bigoplus_{i\in\mathbb{Z}} P_i(X_i)_\bullet$) where $X_i \in \mathcal{A}$, from which we immediately see that the $\mathcal{E}$-injective and $\mathcal{E}$-projective objects coincide.

Given a complex $X_\bullet = (X_i, d_i)$, we let $I(X_\bullet)_\bullet := \bigoplus_{i\in\mathbb{Z}} I_i(X_i)$, and $\Sigma X_\bullet := (X_{i-1}, -d_{i-1}^{X_\bullet})$. We define $\iota_{X_\bullet} : X_\bullet \to I(X_\bullet)_\bullet$ by $\iota_{X_i} : X_i \to X_i \oplus X_{i-1}$ such that $\iota_{X_i} = \begin{pmatrix} \mathrm{id}_{X_i} \\ d_i \end{pmatrix}$, which has the left inverse $(\mathrm{id}_{X_i}, 0)$. Likewise we define $\pi_{X_\bullet} : I(X_\bullet)_\bullet \to \Sigma X_\bullet$ by $\pi_{X_i} : X_i \oplus X_{i-1} \to X_{i-1}$ such that $\pi_{X_i} = (-d_i, \mathrm{id}_{X_{i-1}})$ which has the right inverse $\begin{pmatrix} 0 \\ \mathrm{id}_{X_{i-1}} \end{pmatrix}$. Since $\pi_{X_i}\iota_{X_i} = 0$ we thus have an $\mathcal{E}$-exact sequence:

$$0 \to X_\bullet \xrightarrow{\iota_{X_\bullet}} I(X_\bullet)_\bullet \xrightarrow{\pi_{X_\bullet}} \Sigma X_\bullet \to 0,$$

which exhibits $\iota_{X_\bullet}$ as monic and $\pi_{X_\bullet}$ as epic and which shows that we have enough $\mathcal{E}$-injectives as well as enough $\mathcal{E}$-projectives, since any complex $X_\bullet$ can be realised as $\Sigma Y_\bullet$ for some complex $Y_\bullet$, namely $Y_i = X_{i+1}$ and $d_i^{Y_\bullet} = -d_{i+1}^{X_\bullet}$.

This also shows that all $\mathcal{E}$-injectives are direct summands of $\mathcal{E}$-injective objects of the form $\bigoplus_{i\in\mathbb{Z}} I_i(X_i)_\bullet$ since if $E_\bullet$ is any $\mathcal{E}$-injective object, then $0 \to E_\bullet \to \bigoplus_{i\in\mathbb{Z}} I_i(E_i)_\bullet \to \Sigma E_\bullet \to 0$ splits, so $\bigoplus_{i\in\mathbb{Z}} I_i(E_i)_\bullet \cong E_\bullet \oplus \Sigma E_\bullet$.

All in all we have proved the following proposition:

Proposition 4.2.1 *The category* $(\mathrm{ch}(\mathcal{A}), \mathcal{E})$ *is a Frobenius category.*

Thus we may take the stable Frobenius category of $(\mathrm{ch}(\mathcal{A}), \mathcal{E})$, but, as the following proposition will show, this yields exactly the homotopy category $K(\mathcal{A})$.

Proposition 4.2.2 *Let* $f_\bullet \in \mathrm{Hom}_{\mathrm{ch}(\mathcal{A})}(X_\bullet, Y_\bullet)$*. Then* $f_\bullet$ *factors through an* $\mathcal{E}$*-injective (i.e.* $f \sim_i 0$*) if and only if* $f_\bullet$ *is null-homotopic.*

Proof. Assume $f_\bullet$ factors through an $\mathcal{E}$-injective $I_\bullet$ as $f_\bullet = \pi_\bullet\iota_\bullet : X_\bullet \to I_\bullet \to Y_\bullet$. We may assume that $I_\bullet = \bigoplus_{i\in\mathbb{Z}} I_i(M_i)$ for some $M_i \in \mathcal{A}$ with differential $d_i^I : M_i \oplus M_{i-1} \to M_{i-1} \oplus M_{i-2}$ given by $d_i^I = \begin{pmatrix} 0 & \mathrm{id}_{M_{i-1}} \\ 0 & 0 \end{pmatrix}$, since if $f_\bullet$ factors over an arbitrary injective object $E_\bullet$, then it factors over the sum $E_\bullet \oplus \Sigma E_\bullet \cong \bigoplus_{i\in\mathbb{Z}} I_i(M_i)$.

Note that in order for $\iota_i = \binom{a_i}{b_i}$ and $\pi_i = (v_i, w_i)$ to be chain maps (i.e. commute with the differentials) the following equalities must hold:

$$a_{i-1}d_i^X = b_i, \qquad b_{i-1}d_i^X = 0, \qquad d_i^Y w_i = v_{i-1}, \qquad d_i^Y v_i = 0.$$

Let $b_i^I : M_{i-1} \oplus M_{i-2} \to M_i \oplus M_{i-1}$ be given by $\begin{pmatrix} 0 & 0 \\ \mathrm{id}_{M_{i-1}} & 0 \end{pmatrix}$. Then by letting $h_i : X_i \to Y_{i+1}$ be given by $h_i = \pi_{i+1} b_{i+1}^I \iota_i = w_{i+1} a_i$ we see that

$$d_{i+1}^Y h_i + h_{i-1} d_i^X = d_{i+1}^Y w_{i+1} a_i + w_i a_{i-1} d_i^X = v_i a_i + w_i b_i = \pi_i \iota_i = f_i.$$

Hence f is null-homotopic.

Conversely, assume $f_\bullet : X_\bullet \to Y_\bullet$ is null-homotopic, i.e. $f_i = d_{i+1}^Y h_i + h_{i-1} d_i^X$, where $h_i : X_i \to Y_{i+1}$. Then $f_\bullet$ factors through $P(Y_\bullet)_\bullet := \bigoplus_{i\in\mathbb{Z}} P_i(Y_i)$ as

$$\begin{array}{ccccc} X_i & \xrightarrow{\iota_i} & Y_{i+1} \oplus Y_i & \xrightarrow{\pi_i} & Y_i \\ \downarrow{\scriptstyle d_i^X} & & \downarrow{\scriptstyle \begin{pmatrix} 0 & \mathrm{id}_{Y_i} \\ 0 & 0 \end{pmatrix}} & & \downarrow{\scriptstyle d_i^Y} \\ X_{i-1} & \xrightarrow[\iota_{i-1}]{} & Y_i \oplus Y_{i-1} & \xrightarrow[\pi_{i-1}]{} & Y_{i-1} \end{array}$$

where $\iota_i = \begin{pmatrix} h_i \\ h_{i-1} d_i^X \end{pmatrix}$ and $\pi_i = (d_{i+1}^Y, \mathrm{id}_{Y_i})$. Indeed, the diagram commutes and most importantly we have that $\pi_i \iota_i = d_{i+1}^Y h_i + h_{i-1} d_i^X = f_i$, as desired. □

Thus $\underline{\mathrm{ch}}(\mathcal{A}) = K(\mathcal{A})$, which means that $K(\mathcal{A})$ is triangulated! Note that the shift functor on an object $X_\bullet$ is usually denoted by $X_\bullet[1]$. We will use this notation from this point forward.

With this definition of $K(\mathcal{A})$, we can give a very useful characterization of its cones:

Lemma 4.2.3 Given a triangle $X_\bullet \xrightarrow{f_\bullet} Y_\bullet \to \mathrm{cone}(f_\bullet) \to X_\bullet[1]$ in $K(\mathcal{A})$, the object $\mathrm{cone}(f_\bullet)$ is isomorphic to the object $X_\bullet[1] \oplus Y_\bullet$, with the differentials $d_i^{X_\bullet[1] \oplus Y_\bullet} = \begin{pmatrix} d_i^{X_\bullet[1]} & 0 \\ f_{i-1} & d_i^Y \end{pmatrix}$.

Proof. Recall that $\operatorname{cone}(f_\bullet)$ in a Frobenius category is the pushout of $f_\bullet : X_\bullet \to Y_\bullet$ and $\iota_X : X_\bullet \to I(X_\bullet)_\bullet$. Since pushouts are unique up to isomorphism we simply need to show that $X_\bullet[1] \oplus Y_\bullet$ is a pushout of $f_\bullet$ and ι_X. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_\bullet & \xrightarrow{\iota_X} & I(X_\bullet)_\bullet & \xrightarrow{\pi_X} & X_\bullet[1] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_\bullet} & & \downarrow{\scriptstyle \overline{f}_\bullet} & & \| & & \\
0 & \longrightarrow & Y_\bullet & \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} & X_\bullet[1]\oplus Y_\bullet & \xrightarrow{(1,0)} & X_\bullet[1] & \longrightarrow & 0
\end{array}$$

It is obvious that the rows are $\mathcal{E}$-exact. We see that by letting $\overline{f}_i : X_i \oplus X_{i-1} \to X_{i-1} \oplus Y_i$ be given by $\overline{f}_i = \left(\begin{smallmatrix} -d_i^X & \mathrm{id}_{X_{i-1}} \\ f_i & 0 \end{smallmatrix}\right)$, the diagram commutes. We need to check that $\overline{f}_\bullet$ is a morphism of chain complexes, and indeed, (note, since $f_\bullet$ is a chain map we have $f_{i-1}d_i^X = d_i^Y f_i$)

$$d_i^{X_\bullet[1]\oplus Y_\bullet}\overline{f}_i = \begin{pmatrix} -d_{i-1}^X & 0 \\ f_{i-1} & d_i^Y \end{pmatrix}\begin{pmatrix} -d_i^X & \mathrm{id}_{X_{i-1}} \\ f_i & 0 \end{pmatrix} = \begin{pmatrix} (-d_{i-1}^X)(-d_i^X) & -d_{i-1}^X \\ f_{i-1}(-d_i^X)+d_i^Y f_i & f_{i-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -d_{i-1}^X \\ 0 & f_{i-1} \end{pmatrix} = \begin{pmatrix} -d_{i-1}^X & \mathrm{id}_{X_{i-2}} \\ f_{i-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{id}_{X_{i-1}} \\ 0 & 0 \end{pmatrix} = \overline{f}_{i-1}d_i^{I(X_\bullet)_\bullet}.$$

Finally, since the cokernels of ι_X and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ are isomorphic, it follows from Proposition 0.1.6 that the left square of the diagram is a pushout square, as desired. □

This justifies the notation $\operatorname{cone}(f_\bullet)$, since this is exactly the mapping cone from homological algebra. One can then see that this triangulated structure coincides with the one presented in [HJ10].

These next lemmas will aid us in achieving a rigorous definition of the derived category.

Lemma 4.2.4 *If $f_\bullet$ and $g_\bullet$ are homotopic, then they induce the same map on homology. In particular, the homology functors $H_i : K(\mathcal{A}) \to \mathcal{A}$ are well defined on $K(\mathcal{A})$.*

Proof. By Proposition 4.2.2, $f_\bullet - g_\bullet$ factors over an injective object which is acyclic. Hence the $f_\bullet - g_\bullet$ induces the zero map on homology, and since H_i is additive we get $H_i(f_\bullet - g_\bullet) = H_i(f_\bullet) - H_i(g_\bullet) = 0$, i.e. $H_i(f_\bullet) = H_i(g_\bullet)$. □

Hence this next lemma holds true in both $\operatorname{ch}(\mathcal{A})$ and $K(\mathcal{A})$.

Lemma 4.2.5 *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in* $\operatorname{ch}(\mathcal{A})$ *(or in $K(\mathcal{A})$). Then $f_\bullet$ is a quasi-isomorphism if and only if* $\operatorname{cone}(f)$ *is acyclic.*

Proof. We have a natural short exact sequence $0 \to Y_\bullet \xrightarrow{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} \operatorname{cone}(f) \xrightarrow{(1,0)} X_\bullet[1] \to 0$ in $\operatorname{ch}(\mathcal{A})$ (or $K(\mathcal{A})$) which induces a long exact sequence of homology:

$$\cdots \to H_{i+1}(Y_\bullet) \to H_{i+1}(\operatorname{cone}(f)) \to H_i(X_\bullet) \to H_i(Y_\bullet) \to H_i(\operatorname{cone}(f)) \to \dots,$$

from which the desired result follows. □

4.3 Triangulation of the derived and relatively derived category

With the homotopy category investigated, we can use this information to define the derived category. To this end we need the following lemma.

Lemma 4.3.1 *The full subcategory $\mathcal{N}(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic chain complexes is a triangulated subcategory of $K(\mathcal{A})$. Furthermore, the collection* $\mathrm{Mor}_{\mathcal{N}(\mathcal{A})}$ *consists of all quasi-isomorphisms.*

Proof. It is immediate that $\mathcal{N}(\mathcal{A})$ is full and isomorphism-closed, and that $\mathcal{N}(\mathcal{A})[1] = \mathcal{N}(\mathcal{A})$. Since $\mathcal{N}(\mathcal{A})$ is full, additivity of $\mathcal{N}(\mathcal{A})$ follows from the fact that 0 is acyclic and that if X, Y are acyclic, then $X \oplus Y$ is also acyclic.

Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a triangle in $K(\mathcal{A})$ with X and Y acyclic. Then f must be a quasi-isomorphism and hence, by Lemma 4.2.5, $\mathrm{cone}(f)$ is acyclic.

It remains to show that $\mathrm{Mor}_{\mathcal{N}(A)}$ consists of all quasi-isomorphisms, but this fact follows immediately from Lemma 4.2.5. □

Thus the following definition of the derived category as the Verdier localisation of $K(\mathcal{A})$ by $\mathcal{N}(\mathcal{A})$ is well-defined and shows that $D(\mathcal{A})$ is triangulated.

Definition 4.3.2 *The derived category of an abelian category $\mathcal{A}$ is defined as the Verdier localisation $D(\mathcal{A}) := K(\mathcal{A})/\mathcal{N}(\mathcal{A})$.*

It is easy to see that this definition coincides with one of those presented in [HJ10] (pp. 25-26) (namely $\widetilde{D}(\mathcal{A})$). Furthermore, the triangulated structure also coincides with the one presented in [HJ10] (Definition 7.16).

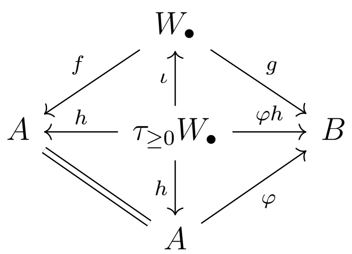
In fact, one can prove that any short exact sequence in $\mathcal{A}$ induces a triangle in $D(\mathcal{A})$, see [HJ10]. This further justifies the idea that triangles play the role of exact sequences in abelian categories, as described in Subsection 1.1.

Next, we prove a well known fact which will be useful for the following theorem.

Proposition 4.3.3 *There exists a fully faithful embedding $\mathcal{A} \to D(\mathcal{A})$.*

Proof. There is a natural fully faithful embedding $E : \mathcal{A} \to K(\mathcal{A})$ sending $A \mapsto [\cdots \to 0 \to A \to 0 \to \cdots]$, so we may consider $\mathcal{A}$ as the image $E(\mathcal{A})$ of this embedding.

We claim that the restricted functor $F_{univ} : E(\mathcal{A}) \to D(\mathcal{A})$ from Verdiers localisation theorem is fully faithful, so $F_{univ}E$ is the desired functor. Indeed, F_{univ} is clearly injective on the Hom sets. For surjectivity, let $A \xleftarrow{f} W_\bullet \xrightarrow{g} B$ be in $\mathrm{Hom}_{D(\mathcal{A})}(A, B)$ (with $A, B \in E(\mathcal{A})$). Then this roof is equivalent to the roof $A \xleftarrow{\mathrm{id}_A} W_\bullet \xrightarrow{\varphi} B$, where $\varphi = H_0(g)H_0(f)^{-1}$ (recall, f is a quasi-isomorphism). Indeed, let $\tau_{\geq 0}W_\bullet$ be defined by $\tau_{\geq 0}W_i = W_i$ for $i > 0$, $\tau_{\geq 0}W_0 = \ker(d_0^W)$ and $\tau_{\geq 0}W_i = 0$ for $i < 0$. Then the diagram



commutes, where ι is the natural inclusion and h is defined by $h_0 : \ker(d_0^W) \to H_0(W_\bullet) \xrightarrow{H_0(f)} A$ and $h_i = 0$ otherwise, both of which are quasi-isomorphisms. □

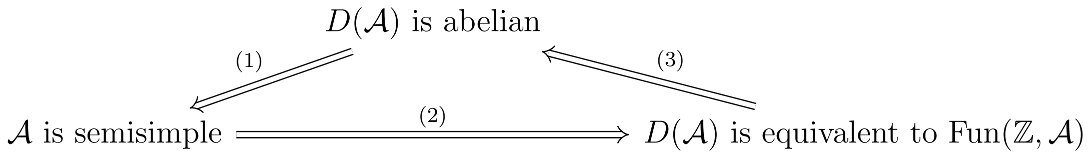
We say that a complex $X_\bullet = (X_i, d_i^X)$ in $K(\mathcal{A})$ (or in $\mathrm{ch}(\mathcal{A})$) is *cyclic* if every differential d_i^X is the zero morphism. Let us write $K_0(\mathcal{A})$ for the full subcategory of $K(\mathcal{A})$ consisting of cyclic complexes. It is easily seen that $K_0(\mathcal{A})$ is just $\mathbb{Z}$ copies of $\mathcal{A}$, so it is equivalent to $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$, where we consider $\mathbb{Z}$ as a discrete category on the underlying set (i.e. no morphisms, except the identity). Let $F_h : K(\mathcal{A}) \to K_0(\mathcal{A})$ be the functor defined by $F_h((X_i, d_i^X)) = (H_i(X_\bullet), 0)$ and $F_h(f : X_\bullet \to Y_\bullet) = H_i(f)_\bullet$. Since F_h maps quasi-isomorphisms to isomorphisms, by Verdiers localisation theorem, F_h factors through $D(\mathcal{A})$ as $K(\mathcal{A}) \xrightarrow{F_u} D(\mathcal{A}) \xrightarrow{F_k} K_0(\mathcal{A})$.

Theorem 4.3.4 *Let $\mathcal{A}$ be an abelian category. The following are equivalent:*

- $D(\mathcal{A})$ is abelian
- $\mathcal{A}$ is semisimple
- $D(\mathcal{A})$ is equivalent to $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$, where we consider $\mathbb{Z}$ as a discrete category.

In the following proof, the implication (2) is due to [IM88] with only minor changes to fit our course of action. The general composition of the proof is due to [b)].

Proof. The plan for the proof is showing the following implications:



(1) Firstly, assume that $D(\mathcal{A})$ is abelian. Then by Theorem 1.2.8, $D(\mathcal{A})$ is semisimple, which means that every morphism f has a pseudo-inverse. By Proposition 4.3.3, there is a fully faithful embedding of $\mathcal{A}$ into $D(\mathcal{A})$, and it therefore follows that every morphism f' in $\mathcal{A}$ has a pseudo-inverse, which means that $\mathcal{A}$ is semisimple.

(2) Secondly, assume that $\mathcal{A}$ is semisimple. We will show $D(\mathcal{A})$ is equivalent to $K_0(\mathcal{A})$ (which is equivalent to $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$), by showing that the functor F_k, defined above, is an equivalence of categories. Let $X_\bullet = (X_i, d_i^X)$ be any complex in $K(\mathcal{A})$ and set $B_i = \mathrm{Im}(d_{i+1}^X)$ and $Z_i = \ker(d_i^X)$. Then we have two natural short exact sequences in $\mathcal{A}$:

$$0 \to Z_i \to X_i \to B_{i-1} \to 0,$$

$$0 \to B_i \to Z_i \to H_i(X_\bullet) \to 0.$$

Since $\mathcal{A}$ is semisimple, we therefore have that $(X_i, d_i^X) \cong (B_i \oplus H_i(X_\bullet) \oplus B_{i-1}, d_i^{B_i \oplus H_i(X_\bullet) \oplus B_{i-1}})$, where $d_i^{B_i \oplus H_i(X_\bullet) \oplus B_{i-1}} = \begin{pmatrix} 0 & 0 & \mathrm{id}_{B_{i-1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We let X_i represent $B_i \oplus H_i(X_\bullet) \oplus B_{i-1}$.

Thus we can see that $f^X : (X_i, d_i^X) \to (H_i(X_\bullet), 0)$ given by $f_i^X = (0, \mathrm{id}_{H_i(X)}, 0)$ and $g^X : (H_i(X), 0) \to (X^i, d^i)$ given by $g_i^X = \begin{pmatrix} 0 \\ \mathrm{id}_{H_i(X)} \\ 0 \end{pmatrix}$ are morphisms in $K(\mathcal{A})$.

Let $F_\ell : K_0(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ be defined as the composition of the embedding $F_\iota : K_0(\mathcal{A}) \to K(\mathcal{A})$ with the functor F_u (defined above). It is clear that $F_k F_\ell = F_h F_\iota$ is naturally isomorphic to $\mathrm{id}_{K_0(\mathcal{A})}$. Conversely, $F_\ell F_k : D(\mathcal{A}) \to D(\mathcal{A})$ maps (X_i, d_i^A) to $(H_i(X_\bullet), 0)$. Since $f^X : X_\bullet \to F_\ell(F_k(X_\bullet))$ is an isomorphism with inverse g^X in $D(\mathcal{A})$, it follows that $\{f^X\}_{X\in\mathcal{D}(\mathcal{A})}$ is a natural isomorphism between $\mathrm{id}_{D(\mathcal{A})}$ and $F_\ell F_k$, with inverse $\{g^X\}_{X\in\mathcal{D}(\mathcal{A})}$.

(3) Lastly, if $D(\mathcal{A})$ is equivalent to $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$, then obviously $D(\mathcal{A})$ is abelian since $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ is abelian. □

Example 4.3.5 The category of finite-dimensional vector spaces $\mathbf{FinVect}_k$ over a field k is semisimple. Hence $D(\mathbf{FinVect}_k)$ is equivalent to $\mathrm{Fun}(\mathbb{Z}, \mathbf{FinVect}_k)$ and is abelian.

We finish off this paper by giving a quick and dirty way of defining the relative derived category, and consequently the Gorenstein derived category.

Let $\mathcal{X}$ be a subcategory of the abelian category $\mathcal{A}$. We say a complex $A_\bullet \in \mathrm{ch}(\mathcal{A})$ is *$\mathcal{X}$-acyclic* if the induced chain complex $\mathrm{Hom}_{\mathcal{A}}(X, A_\bullet)$ in $\mathrm{ch}(\mathbf{Ab})$ is acyclic for any $X \in \mathcal{X}$. A morphism $f_\bullet : A_\bullet \to B_\bullet$ in $\mathrm{ch}(\mathcal{A})$ is called an *$\mathcal{X}$-quasi-isomorphism* if the induced map $\mathrm{Hom}_{\mathcal{A}}(X, f_\bullet) : \mathrm{Hom}_{\mathcal{A}}(X, A_\bullet) \to \mathrm{Hom}_{\mathcal{A}}(X, B_\bullet)$ is a quasi-isomorphism in $\mathrm{ch}(\mathbf{Ab})$ for any $X \in \mathcal{X}$.

Lemma 4.3.6 *Let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism in $K(\mathcal{A})$ (or $\mathrm{ch}(\mathcal{A})$). Then $f_\bullet$ is an $\mathcal{X}$-quasi-isomorphism if and only if $\mathrm{cone}(f)$ is $\mathcal{X}$-acyclic.*

Proof. Let $X \in \mathcal{X}$. The natural exact sequence $0 \to B_\bullet \to \mathrm{cone}(f) \to A_\bullet[1] \to 0$ is degree-wise split exact, so by applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ to it we will obtain another short exact sequence $0 \to \mathrm{Hom}_{\mathcal{A}}(X, B_\bullet) \to \mathrm{Hom}_{\mathcal{A}}(X, \mathrm{cone}(f)) \to \mathrm{Hom}_{\mathcal{A}}(X, A_\bullet) \to 0$ in $\mathrm{ch}(\mathbf{Ab})$ which induces a long exact sequence of homology:

$$\cdots \to H_i(\mathrm{Hom}_{\mathcal{A}}(X, A_\bullet)) \to H_i(\mathrm{Hom}_{\mathcal{A}}(X, B_\bullet)) \to H_i(\mathrm{Hom}_{\mathcal{A}}(X, \mathrm{cone}(f))) \to \dots,$$

from which the desired result follows. □

Proposition 4.3.7 *Let $K_{\mathcal{X}}(\mathcal{A})$ denote the full subcategory of $K(\mathcal{A})$ consisting of $\mathcal{X}$-acyclic complexes. Then $K_{\mathcal{X}}(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$ and $\mathrm{Mor}_{K_{\mathcal{X}}(\mathcal{A})}$ consists of all $\mathcal{X}$-quasi-isomorphisms.*

Proof. It is immediate that $K_{\mathcal{X}}(\mathcal{A})$ is an additive (the Hom functor is additive), full and isomorphism-closed subcategory, and that $K_{\mathcal{X}}(\mathcal{A})[1] = K_{\mathcal{X}}(\mathcal{A})$.

It follows from Lemma 4.3.6 that if $A_\bullet \to B_\bullet \to C_\bullet \to X_\bullet[1]$ is a triangle in $K(\mathcal{A})$ with $A_\bullet, B_\bullet \in K_{\mathcal{X}}(\mathcal{A})$, then $C_\bullet \in K_{\mathcal{X}}(\mathcal{A})$.

Lastly, the fact that $\mathrm{Mor}_{K_{\mathcal{X}}(\mathcal{A})}$ consists of all $\mathcal{X}$-quasi-isomorphisms is immediate from Lemma 4.3.6. □

Thus we may use Verdiers localisation theorem to obtain the $\mathcal{X}$-relative derived category of $\mathcal{A}$, denoted $D_{\mathcal{X}}(\mathcal{A})$.

Definition 4.3.8 *The* $\mathcal{X}$-relative derived category *$D_{\mathcal{X}}(\mathcal{A})$ of $\mathcal{A}$ is defined as the Verdier localisation, $D_{\mathcal{X}}(\mathcal{A}) := K(\mathcal{A})/K_{\mathcal{X}}(\mathcal{A})$.*

Remark 4.3.9 If $\mathcal{A}$ has enough projectives and if we let $\mathcal{X}$ be the class of all projective objects, then one can show that $A_\bullet$ is $\mathcal{X}$-acyclic if and only if $A_\bullet$ is acyclic in the usual sense (since $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is then an exact functor). Thus, in this case we recover the derived category $D(\mathcal{A}) = K(\mathcal{A})/K_{\mathcal{X}}(\mathcal{A}) = K(\mathcal{A})/\mathcal{N}(A)$ (see, [Di+14]).

An important example of a relative derived category is the Gorenstein derived category. For this we need the notion of Gorenstein projective complexes.

Definition 4.3.10 *An acyclic complex of projective objects $P_\bullet$ in $\text{ch}(\mathcal{A})$ is called* totally acyclic *if the induced complex $\text{Hom}(P_\bullet, Q)$ is acyclic for any projective object Q.*

An object G in $\mathcal{A}$ is called Gorenstein projective *if there exists a totally acyclic complex of projectives $P_\bullet$ such that $\text{Im}(P_i \to P_{i-1}) \cong G$ for some $i \in \mathbb{Z}$.*

We denote the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects as $\mathcal{GP}$-$\mathcal{A}$.

We have a dual notion of Gorenstein *injective* objects and a full subcategory $\mathcal{GI}$-$\mathcal{A}$ consisting of these objects. However we will only consider $\mathcal{GP}$-$\mathcal{A}$.

One may consider the $\mathcal{GP}$-relative derived category which leads us to the definition of the Gorenstein derived category:

Definition 4.3.11 *The* Gorenstein derived category $D_{\mathcal{GP}}(\mathcal{A})$ *of $\mathcal{A}$ is the $\mathcal{GP}$-relative derived category of $\mathcal{A}$.*

Example 4.3.13 If all objects of $\mathcal{A}$ has finite projective dimension (i.e. their projective resolution has finite length) then the Gorenstein projective objects and the regular projective objects coincide (see, [GZ10]).

Remark 4.3.14 Example 4.3.13 and Remark 4.3.9 tells us that if every object of $\mathcal{A}$ has finite projective dimension, then $D(\mathcal{A}) \cong D_{\mathcal{GP}}(\mathcal{A})$. In particular, if $\mathcal{A}$ is semisimple, we have $D(\mathcal{A}) \cong D_{\mathcal{GP}}(\mathcal{A})$, since every object has projective dimension 0.

However, this next proposition holds true for any abelian category $\mathcal{A}$.

Proposition 4.3.14 *There exists a fully faithful embedding $\mathcal{A} \to D_{\mathcal{GP}}(\mathcal{A})$.*

Proof. The proof is analogous to that of Proposition 4.3.3 and can be found in [GZ10]. □

Hence we can prove the Gorenstein version of Theorem 4.3.4:

Theorem 4.3.15 *The Gorenstein derived category $D_{\mathcal{GP}}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple. In this case we also have that $D_{\mathcal{GP}}(\mathcal{A}) \cong \text{Fun}(\mathbb{Z}, \mathcal{A})$.*

Proof. Assume $D_{\mathcal{GP}}(\mathcal{A})$ is abelian. Then $D_{\mathcal{GP}}(\mathcal{A})$ is semisimple by Theorem 1.2.8, and every morphism f has a pseudo-inverse. By Proposition 4.3.14, there is a fully faithful embedding $\mathcal{A} \to D_{\mathcal{GP}}(\mathcal{A})$, and it therefore follows that every morphism f' in $\mathcal{A}$ has a pseudo-inverse. Hence $\mathcal{A}$ is semisimple.

Conversely, assume $\mathcal{A}$ is semisimple. Then $D_{\mathcal{GP}}(\mathcal{A}) \cong D(\mathcal{A}) \cong \text{Fun}(\mathbb{Z}, \mathcal{A})$, by Remark 4.3.15 and Theorem 4.3.4. □

In other words, abelian Gorenstein derived categories are pretty boring.

4.4 Final remarks

We have only just scratched the surface of the theory of triangulated categories, but we can already see its power. With a bit of effort, we were able to give the important categories $K(\mathcal{A})$ and $D(\mathcal{A})$ a nice structure, which works as a framework to aid in further studies of these (any many more) categories (e.g. Theorem 4.3.4). Futhermore, with minimal effort, we were able to generalize the derived category to the relative derived category.

We refer the interested reader to [Nee01] for more general theory of triangulated categories.

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