Riemann-Roch on graphs, debt games and applications.

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This is a brief survey of two papers.[1][2] We cover the nature of divisors on graphs and show part of the proof of a graph-theoretic counterpart of the classic Riemann-Roch theorem. Furthermore, the theory is applied to give a new volumetric version of Kirchoff's theorem.

1 DOLLAR DEBT GAME

Consider any connected graph, G, with an integer weight at each vertex, $v \in V(G)$. Think of the weight as the amount of dollars (or debt) the vertex has. For any vertex, v, we allow two legal moves: Either v takes a dollar from each neighbor, w (i.e. v, w connected by an edge $e = vw \in E(G)$), trough each edge. Or v gives a dollar to each neighbor, through each edge. The goal of the game is then to get every vertex out of debt. The natural question is of course; for which configurations is the game winnable?

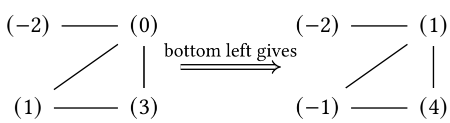

Figure 1: Example of legal move.

First we need some notation. We denote the genus of any graph, G, by $g := |E(G)| - |V(G)| + 1$ and define the degree, $\deg(D)$, of a vertex-weighted graph, D, as the sum of all the weights (i.e. the total number of dollars). Then we have the following:

Theorem 1.1. If $\deg(D) \geq g$, then the game is always winnable. Furthermore, if $\deg(D) \leq g-1$, then there exists some configuration for which the game is *not* winnable.

We also define $r(D)$ to be -1 if the game is not winnable and to be a non-negative integer, k, denoting the largest number of dollars we can remove from the game (in any way), so that it remains winnable.

Finally, we define the canonical configuration K to be given by $K(v) = \deg(v) - 2$, where $\deg(v)$, with $v \in V(G)$, is the number of edges incident to v.

We can now express the Riemann-Roch theorem for graphs:

Theorem 1.2. (Riemann-Roch for graphs) For any configuration D on any graph G we have

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

An immediate corollary of this is that if $\deg(D) = g - 1$, then $r(D) = r(K - D)$, and we see that D is winnable if and only if $K - D$ is winnable.

It is easy to see how Theorem 1.1. follows from Theorem 1.2., but we will see later that these theorems can be expressed in a more interesting context!

[1] arXiv:math/0608360v3 [math.CO] 9 Jul 2007
[2] arXiv:1304.4259v2 [math.CO] 24 Sep 2014

2 RIEMANN-ROCH FOR GRAPHS

Let G be any graph without loop edges, and let Q denote the Laplacian matrix. We let the *divisor group*, $\mathrm{Div}(G)$, be the free abelian group on $V(G)$, and write an element $D \in \mathrm{Div}(G)$ as the formal sum $\sum_{v \in V(G)} a_v(v)$, and denote a_v by $D(v)$. Think of D as a configuration on G from the debt game. $\mathrm{Div}(G)$ is the graph analogy to the divisors on a Riemann surface, but we won't go more in depth with the analogies with Riemann surfaces.

$\mathrm{Div}(G)$ has a partial ordering given by $D \geq D'$ if and only if $D(v) \geq D'(v)$ for all $v \in V(G)$. We call a divisor, E, effective if $E \geq 0$ and denote by $\mathrm{Div}_+(G)$ the set of effective divisors.

Furthermore, we define the degree function $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ by $\deg(D) = \sum_{v \in V(G)} D(v)$, i.e. the sum of coefficients of D.

Let $\mathcal{M}(G) = \mathrm{Hom}(V(G), \mathbb{Z})$, the abelian group of integer-valued functions. The Laplacian operator $\Delta : \mathcal{M}(G) \to \mathrm{Div}(G)$ is given by

$$\Delta(f) = \sum_{v \in V(G)} \Delta_v(f)(v)$$

$$\Delta_v(f) = \sum_{e \in E_v} (f(v) - f(w))$$

In fact, one can easily see then that $[\Delta(f)] = Q[f]$.

We define $\mathrm{Div}^k(G) = \{D \in \mathrm{Div}(G) \mid \deg(D) = k\}$ as well as $\mathrm{Div}_+^k(G) := \mathrm{Div}_+(G) \cap \mathrm{Div}^k(G)$.

We can now define the principal divisors $\mathrm{Prin}(G) := \Delta(\mathcal{M}(G))$ and note that this must specially be a subgroup of $\mathrm{Div}^0(G)$. And so, we can now finally define the quite interesting quotient groups, the *Picard groups* of G:

$$\mathrm{Pic}^n(G) = \mathrm{Div}^n(G)/\mathrm{Prin}(G).$$

and write $[D]$ for the class of in $\mathrm{Pic}^n(G)$ of $D \in \mathrm{Div}^g(G)$.

The Picard groups, have many interesting properties. For starters, the order $\mathrm{Pic}^0(G)$ is the number of spanning trees in G.

We now fix a base vertex v_0 and define the Abel-Jacobi map

$$S_{v_0} : G \to \mathrm{Pic}^0(G), \quad S_{v_0}(v) = [(v) - (v_0)]$$

and for $k \geq 0$ a map $S_{v_0}^{(k)} : \mathrm{Div}_+^k(G) \to \mathrm{Pic}^0(G)$ given by

$$S_{v_0}^{(k)}((v_1) + \cdots + (v_k)) = S_{v_0}(v_1) + S_{v_0}(v_2) + \cdots + S_{v_0}(v_k).$$

then we have

Theorem 2.1. The map, $S^{(k)}$, is surjective if and only if $k \geq g$.

We define a linear equivalence relation $D \sim D'$ if $D - D' \in \mathrm{Prin}(G)$. (In terms of dollar-debt game $D \sim D'$ if and only if there is a series of moves that connect the two). Note that it follows that D and D' have same degree and that Pic^n is the set of equivalence classes of degree n divisors on G.

One can also show that $S^{(k)}$ is surjective if and only if every divisor of degree k is linearly equivalent to an effective divisor. From this we see that Theorem 2.1 is equivalent to Theorem 1.1!

Let

$$|D| := \{E \in \mathrm{Div}(G) \mid E \geq 0, E \sim D\}$$

which we use to define the dimension as $r(D) = -1$ if $|D| = \emptyset$ and $r(D) = \max\{ s \mid |D - E| \neq \emptyset \text{ for all } E \text{ of degree } s\}$. Note $r(D)$ is invariant under the equivalence, $\sim$. (In terms of dollar-debt game $r(D)$ is the maximum number of dollars you can remove (in any way) from the board such that it remains winnable).

Finally, the canonical divisor is given by

$$K = \sum_{v \in V(G)} (\deg(v) - 2)(v).$$

Note $\deg(K) = 2|E(G)| - 2|V(G)| = 2g - 2$, since we get every edge twice.

Thus we can restate Theorem 1.2. in this more rigorous framework

Theorem 2.2. (Riemann-Roch for graphs) Let G a graph, D a divisor on G. Then

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

The properties of Riemann-Roch for Graphs has a lot of overlap with those of Riemann-Roch for Riemann surfaces, however it's not one to one and one must be careful.

An easy consequence of this is Clifford's Theorem for Graphs:

Corollary 2.3. Let D be an effective divisor such that $|K - D| \neq \emptyset$ (called special) on G. Then

$$r(D) \leq \frac{1}{2}\deg(D)$$

3 THE PROOF

There was nothing special about the set, $V(G)$, on which defined $\mathrm{Div}(G)$, so we could've used any set X instead of $V(G)$. Likewise the effective divisors and Div_+^d did not depend on anything other then the set structure, and can be generalized to any set, X. Further, if we can define $\sim$ on $\mathrm{Div}(X)$ satisfying:

(E1) If $D \sim D'$ then $\deg(D) = \deg(G')$.

(E2) If $D_1 \sim D_1'$ and $D_2 \sim D_2'$, then $D_1 + D_2 \sim D_1' + D_2'$.

Then we can define $|D| := \{E \in \mathrm{Div}(X) \mid E \geq 0, E \sim D\}$, and $r : \mathrm{Div}(X) \to \{-1, 0, 1, 2, ...\}$ in the same way we did for graphs.

Finally let $\mathcal{N} = \{D \in \mathrm{Div}(X) \mid \deg(D) = g - 1, |D| = \emptyset\}$, and K some divisor with degree $2g - 2$.

Then we have the following generalization of Riemann-Roch, which we will take for granted:

Theorem 3.1. The Riemann-Roch equality,

$$r(D) - r(K - D) = \deg(D) + 1 - g,$$

holds for all $D \in \mathrm{Div}(G)$ iff the following two properties hold:

(RR1) For every $D \in \mathrm{Div}(X)$, there is $\nu \in \mathcal{N}$ such that either $|D| = \emptyset$ or $|\nu - D| = \emptyset$, but never both.

(RR2) For every $D \in \mathrm{Div}(X)$ with $\deg(D) = g - 1$, either both $|D| = \emptyset$ and $|K - D| = \emptyset$, or both are non-empty.

To prove this one needs v_0-*reduced* divisors which are divisors which satisfy $D(v) \geq 0$ for all $v \neq v_0$, and that for every non-empty subset $A \subseteq V(G) - \{v_0\}$ there is $v \in A$ such that $D(v) < \mathrm{outdeg}_A(v)$, where outdeg_A is the number of edges from v which end not in A.

In terms of dollar-debt game: v_0 is the only vertex which can be in debt and if all $v \in A$ were to make a lending move, some vertex of A would go into debt, for all $A \subset V(G) - \{v_0\}$.

Proposition 3.2. Fix v_0 then for every divisor D there is a unique v_0-reduced divisor D' such that $D \sim D'$.

Given some total order $<_P$ on $V(G)$ we define a specefic divisor:

$$\nu_P = \sum_{v \in V(G)} (|\{e = vw \in E(G) \mid w <_P v\}| - 1)(v)$$

And notice that $\deg(\nu_P) = |E(G)| - |V(G)| = g - 1$, since we will see every edge and we subtract 1 at each vertex.

It turns out that $\nu_P \in \mathcal{N}$ for all total orders, $<_P$. Which will help us prove the following:

Theorem 3.3. For all divisors D exactly one of the following hold:

(N1) $r(D) \geq 0$

(N2) $r(\nu_P - D) \geq 0$ for some $<_P$.

Proof. Fix v_0. We may assume D is v_0-reduced by prop. 3.2. We define an order $v_1, v_2, ..., v_{|V(G)|-1}$ (i.e. $v_i <_P v_j$ iff $i < j$) iductively: If $v_0, ..., v_{k-1}$ defined then let $A_k = V(G) - \{v_0, ..., v_{k-1}\}$ and choose v_k so that $D(v_k) < \mathrm{outdeg}_{A_k}(v_k)$.

Now for every $v_k \neq v_0$ we have by definition of ν_P

$$\begin{aligned} D(v_k) &\leq \mathrm{outdeg}_{A_k} - 1 \\ &= |\{e = v_k v_j \mid v_j < v_k\}| - 1 \\ &= \nu_P(v_k). \end{aligned}$$

If $D(v_0) \geq 0$ then $D \geq 0$ (since it's v_0-reduced) and (N1) holds. And if $D(v_0) \leq -1$ then $D \leq \nu_P$, so $\nu_P - D \geq 0$ and (N2) holds. If both $r(D) \geq 0$ and $r(\nu_P - D) \geq 0$ then $r(\nu_P) = r(D + \nu_p - D) \geq r(D) + r(\nu_p - D) \geq 0$, contradicting that $\nu_P \in \mathcal{N}$. □

Corollary 3.4. For all divisors, D, of degree $g - 1$, we have that $D \in \mathcal{N}$ if and only if there exists $<_P$ on $V(G)$ such that $D \sim \nu_P$.

Proof. If $\nu_P - D \sim E$ with $E \geq 0$ then $\deg(E) = \deg(\nu_P - D) = 0$ and so $E = 0$ and $D \sim \nu_P$. □

Finally we can prove Theorem 2.2:

Proof. (of Theorem 2.2) We need to show RR1 and RR2 hold.

Assume $D \in \mathrm{Div}(G)$ with $r(D) \geq 0$. For all $\nu \in \mathcal{N}$ we have $r(\nu - D) = -1$ and so $|D| \neq \emptyset$ and $|\nu - D| = \emptyset$. So RR1 holds.

On the other hand if $r(D) = -1$, then by [3.3], have $r(\nu_P - D) \geq 0$ for some $<_P$, and then $|D| = \emptyset$ and $|\nu_P - D| \neq \emptyset$. Since $\nu_P \in \mathcal{N}$, so RR1 holds.

For RR2 it suffice to show that for all $D \in \mathcal{N}$ we have $K - D \in \mathcal{N}$. By [3.4] we have $D \sim \nu_P$ for some order $<_P$. Define $<_Q$ by $v <_Q w \Leftrightarrow w <_P v$, i.e. the reverse of P. Then for every v we have

$$\begin{aligned} \nu_P(v) + \nu_Q(v) &= |\{e = vw \mid w <_P v\}| - 1 \\ &\quad + |\{e = vw \mid w <_Q v\}| - 1 \\ &= \deg(v) - 2 = K(v). \end{aligned}$$

So $\nu_Q = K - \nu_P$ and so $K - D \sim K - \nu_P = \nu_Q \in \mathcal{N}$ □

4 APPLICATIONS

This theory and the Riemann-Roch for graphs have many applications in various fields. One example is that we can give a profound proof of the classic Kirchhoff's Theorem:

Theorem 4.1 (Kirchoff's Theorem). The number of spanning trees of a graph G is equal to *any* cofactor of the Laplacian of G.

First we must define *break divisors*: A break divisor, D, on a graph, G, is an effective divisor of degree $g(G)$ such that D restricted to any connected subgraph H of G has the property that $\deg(D|_H) \geq g(H)$. It turns out break divisors have a special connection with spanning trees and the picard groups:

Theorem 4.2 Let g be the genus of G. Then every degree g divisor is equivalent to a unique break divisor. Thus the set of break divisors on G is canonically in bijection with $\mathrm{Pic}^g(G)$

Since $|\mathrm{Pic}^g(G)| = |\mathrm{Pic}^0(G)|$ and the size of $\mathrm{Pic}^0(G)$ is exactly the number of spanning trees, we see that the number of break divisors on G equals the number of spanning trees of G!

Finally, before stating the theorem, we need to talk about tropical curves:

A tropical curve (or metric graph), Γ can be obtained from a graph G by assigning an edge-length $\ell(e) \in \mathbb{R}$ to each edge $e \in E(G)$, and identifying e with the obvious line segment of that length.

Divisors on tropical curves are of the form $\sum_{p\in\Gamma} a_p(p)$, with only finitely many $a_p \in \mathbb{Z}$ non-zero and p allowed to be anywhere along any edge.

Let $f : \Gamma \longrightarrow \mathbb{R}$ be any tropical rational function: a piecewise linear function with only finitely many pieces, each having integer slope. A principal divisor, $\mathrm{div}(f)$ is then given as $\mathrm{div}(f) = \sum_{p\in\Gamma} \mathrm{ord}_p(f)(p)$, with $\mathrm{ord}_p(f)$ being minus the sum of the outgoing slopes of f emanating from p.

We define $\mathrm{Pic}^n(\Gamma)$ in exactly the same way as for graphs:

$$\mathrm{Pic}^n(\Gamma) = \mathrm{Div}^n(\Gamma)/\mathrm{Prin}(\Gamma)$$

Now Pic^0 is no longer finite group but rather a real g-dimensional torus.

Similarly to graphs, we can also define break divisors on a tropical curve, Γ. A break divisor on Γ is an effective divisor, D, of degree g such that D restricted to any closed connected subgraph Γ' has degree at least that of the genus of Γ'. Once again it can be shown that there is a bijection between the break divisors of Γ and $\mathrm{Pic}^g(\Gamma)$.

Furthermore, one has that D is a break divisor on Γ if and only if there exists a spanning tree T of G and an enumeration $e_1^\circ, ..., e_g^\circ$ of $\Gamma\backslash T$ such that $D = (p_1) + \cdots + (p_g)$ with each $p_i \in e_i$ (where e_i° are the open edge of e_i (i.e. endpoints removed)). For a tree T, let B_T be the set of all divisors, $(p_1) + \cdots + (p_g)$, defined as above. Finally let $C_T \subset \mathrm{Pic}^g(T)$ be the image of B_T under the map $D \mapsto [D]$, sending D to it's linear equivalence class. Then we have the following:

Theorem 4.3 We have that $\mathrm{Pic}^g(\Gamma) = \bigcup_{T\in\mathcal{T}} C_T$, where $\mathcal{T}$ is the set of all spanning trees of G. Furthermore, each $C_T \subset \mathrm{Pic}^g(\Gamma)$ is a parallelotope with their relative interior disjoint.

What's more there is a natural metric on $\mathrm{Pic}^g(\Gamma)$ for which $\mathrm{vol}(C_T) = \prod_{e\notin T} \ell(e)$ and the volume of $\mathrm{Pic}^G(\Gamma)$ is naturally related to the determinant of the Laplacian of G, from which one can recover Kirchoff's theorem!

5 QUICK EXAMPLE

Let Γ be the metric graph consisting of 2 vertices connected by 3 edges of length, 2, 1 and 2. We can fix a model, G, for Γ in which all edges have length 1:

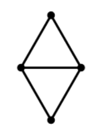

Fig. 1. Model for Γ

This has the following spanning trees:

Fig. 2. Spanning trees of G

And the cell decomposition looks as follows:

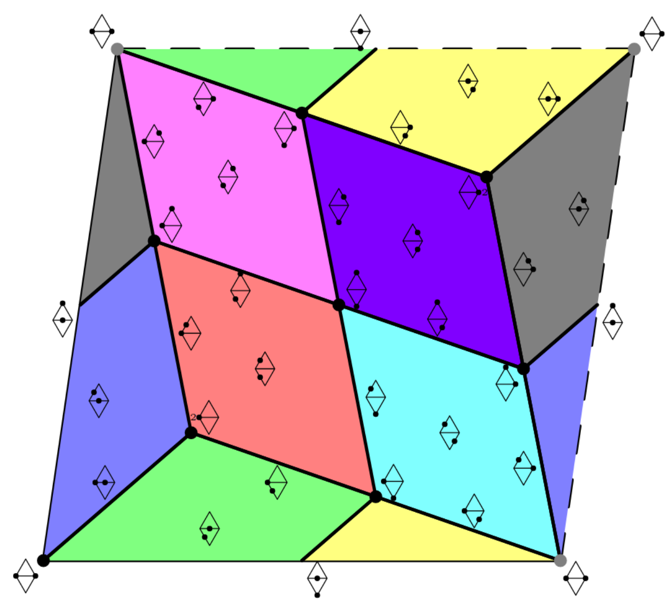

Fig. 3. Cell decomposition