Existance of a closed geodesic on compact Riemannian manifolds

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Introduction

This micro paper is a very brief introduction to the basics of Lyusternik-Schnirelmann theory which concerns closed geodesics:

Definition 1. *A closed geodesic in a Riemannian manifold, M, is a non-constant geodesic segment $c : [0,1] \to M$ such that $c(0) = c(1)$ and $c'(0) = c'(1)$.*

The main theorem of closed geodesics which will be the main target of this paper is stated as follows:

Theorem 1 (Lyusternik-Fet Theorem)**.** *On every compact Riemannian manifold without boundary, there exists a closed geodesic.*

Closed geodesics are of particular interest because closed geodesics of period 1 are exactly the critical points of the energy functional $E : \Lambda M \to \mathbb{R}$ given by:

$$E(c) = \frac{1}{2}\int_0^1 |\dot{c}|^2 dt,$$

where ΛM is the space of smooth 1-periodic curves on M. Note that any closed curve of period p can reparametrized to a critical point via $t \to \gamma(pt)$. We will later see that we make heavy use of the energy functional in the proof of Lyusternik-Fet.

For brevity we omit some of the proofs, but they can be found in Klingenberg [1], which is also the main source of this exposition which we follow closely.

Theory and proof

Let $S = [0,1]/\{0,1\}$. We may define a metric on the space of closed continuous curves, $C^0(S, M)$, on M, by

$$d_\infty(c, c') = \sup_t d(c(t), c'(t))$$

Let $PM \subseteq C^0(S, M)$ be the subspace of piecewise differentiable closed curves. On this we may define the *length functional* and the *energy functional* (as seen before)

$$L(c) = \int_0^1 |\dot{c}| dt \quad \text{and} \quad E(c) = \frac{1}{2}\int_0^1 |\dot{c}|^2 dt,$$

where $c \in PM$. Note that $L(c) \leq \sqrt{2E(c)}$, with equality if $L(c|_{[0,t]}) = tL(c)$ (since in that case c moves with constant velocity 1).

By compactness, there is an $\eta > 0$ such that any p, q with $d(p,q) \leq 2\eta$ can be joined by a unique geodesic, $c_{pq} : [0,1] \to M$ such that $L(c_{pq}) = \sqrt{2E(c_{pq})} = d(p,q)$.

Proposition 1. *Let $\{c_n\}$ be a sequence of piecewise differentiable paths in M with $d(p_n, q_n) \leq \eta$, where $c_n(0) = p_n, c_n(1) = q_n$. If the sequences $\{E(c_n)\}$ and $\{d^2(p_n, q_n)/2\}$ are both convergent with the same limit, then there is a convergent subsequence of $\{c_n\}$ whose limit is a geodesic segment, c, of length equal to $d(c(0), c(1)) \leq \eta$.*

Proof. Omitted. □

We now fix $\kappa > 0$ and choose $k \in 2\mathbb{N}$ such that $4\kappa/k \leq \eta^2$. We have the subspace $P^\kappa M := \{c \in PM \mid E(c) \leq \kappa\}$, and for every $c \in P^\kappa M$ and $t_0 \in S$ we have

$$d^2(c(t_0), c(t_0 + 2/k)) \leq 2E(c),$$

when $2/k \leq \eta^2$.

We now let $0 \leq j \leq k - 2$ be even integer, and for $\sigma \in [j/k, (j + 2)/k]$ we define $\mathcal{D}_\sigma c$ by

$$\mathcal{D}_\sigma c(t) = c(t), \text{ for } t \in [0, j/k] \text{ or } t \in [\sigma, 1]$$

$$\mathcal{D}_\sigma c|_{[j/k,\sigma]} = c_{pq}|_{[j/k,\sigma]}$$

where c_{pq} is the minimizing geodesic from $p = c(j/k)$ to $q = c(\sigma)$, which exists since by construction $d(c(j/k), c(\sigma)) \leq \eta$. Visually we replace a segment of c by the minimizing geodesic between the boundary points of said segment.

To define $\mathcal{D}_\sigma c$ for $\sigma \in [1, 2]$ we let j be as before and take $t \in [1, 1 + 1/k]$ to mean $t \in [0, 1/k]$. Then for $\sigma \in [1 + j/k, 1 + (j + 2)/k]$ we set

$$\mathcal{D}_\sigma c(t) = c(t), \text{ for } t \in [1/k, (j + 1)/k] \text{ or } t \in [\sigma - 1 + 1/k, 1 + 1/k],$$

$$\mathcal{D}_\sigma c|_{[(j+1)/k,\sigma-1+1/k]} = c_{pq}|_{[(j+1)/k,\sigma-1+1/k]}.$$

with $p = c((j + 1)/k), q = c(\sigma - 1 + 1/k)$ and c_{pq} the minimizing geodesic. Again, we replace a segment by a minimizing geodesic.

We then have the following definition and proposition:

Proposition 2. *Define the deformation $\mathcal{D} : [0, 2] \times P^\kappa M \to P^\kappa M$ to be the mapping given by subsequent application, $\mathcal{D}_{2/k}, ..., \mathcal{D}_{2l/k}, \mathcal{D}_\sigma$, of $\mathcal{D}_\sigma$, with l the largest integer such that $2l \leq k\sigma$.*

Then $\mathcal{D}$ is continuous and $E(\mathcal{D}(2, c)) \leq E(c)$, with equality if and only if c is constant or a closed geodesic.

Proof. Omitted. □

Thus $\mathcal{D}$ iteratively replaces segments of the geodesic by minimizing geodesics of between the boundary of the segments.

We now lay out two Lemmas before we attack the main theorem:

Lemma 1. *Let $\{c_n\}$ be a sequence in $P^\kappa M$ such that $\{E(c_n)\}$ and $\{E(\mathcal{D}(2, c_n))\}$ are convergent with same limit, $\kappa_0 > 0$. Then there is a convergent subsequence of $\{c_n\}$ whose limit is a closed geodesic, c_0, with $E(c_0) = \kappa_0$.*

Proof. Choose k as before so that $\mathcal{D}$ is defined. By construction $\mathcal{D}(2, c_n)$ consists of the $k/2$ geodesic segments $\mathcal{D}c_n|_{[(j+1)/k,(j+3)/k]}$ for $j = 0, 2, ..., k-2$.

Applying proposition 1 to the sequence $\{\mathcal{D}(2, c_n)\}$ we obstain a subsequence with limit, c_0, which consists of $k/2$ geodesic segments. Proposition 1 also implies $\{c_n\}$ converges to the same c_0, and so by assumption $E(\mathcal{D}(2, c_0)) = E(c_0) = \kappa_0 > 0$, which by proposition 2 implies that c_0 is a closed geodesic (and not constant since $\kappa_0 \neq 0$). □

Lemma 2. *Let $\kappa_0 > 0$ and $\mathcal{U} \subset P^{\kappa_0}M$ an open neighborhood of the set, C, of closed geodesics whose energy functional equals κ_0. If $C = \emptyset$ we may let $\mathcal{U} = \emptyset$.*

Let $\kappa > \kappa_0$, then there exists ε (dependent on $\mathcal{U}$) such that $\mathcal{D}(2, P^{\kappa_0+\varepsilon}M) \subseteq \mathcal{U} \cup P^{\kappa_0-\varepsilon}M$.

Proof. Clearly $\mathcal{D}(2, -)|_C = \text{id}$ and since $\mathcal{D}$ continuous, there is an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of C such that $\mathcal{D}(\mathcal{U}', 2) \subset \mathcal{U}$. Assume for contradiction ε didn't exist. Then there would exist a sequence $\{c_n\}$ with $c_n \notin \mathcal{U}'$ and

$$\kappa_0 - \tfrac{1}{n} \leq E(\mathcal{D}(2, c_n)) \leq E(c_n) \leq \kappa + \tfrac{1}{n}$$

which means, by lemma 1, that $\{c_n\}$ has a subsequence which converges to a closed geodesic, c_0, with $E(c_0) = \kappa_0$. But since $c_n \notin \mathcal{U}'$ we have $c_0 \notin \mathcal{U}'$, which is a contradiction since c_0 is a closed geodesic and so contained in $C \subset \mathcal{U}'$.

If there is no closed geodesic, c such that $E(c) = \kappa_0$, then Lemma 1 tells us that there is a $\lambda > 0$ such that there is no closed geodesic, c', such that $E(c') \in [\kappa_0 - \lambda, \kappa_o + \lambda]$. □

Theorem 2 (Lyusternik-Fet Theorem)**.** *On every compact Riemannian manifold without boundary, M, there exists a closed geodesic.*

Proof. First assume the first fundamental group $\pi_1(M)$ is non-zero. Then we may choose a non-contractable $c \in PM$. Let $P'M$ denote the space consisting of $c' \in PM$ which are freely homotopic to c. Set $\kappa' = \inf E|_{P'M}$. We then have that $\kappa' > 0$. Indeed, assume for contradiction $\kappa' = 0$ then since $E(c')$ is bounded by κ' we would have $L(c') \leq \sqrt{2E(c')} < \eta$. I.e. c' is a closed curve of length $< \eta$, which is, by choice of η, always contractable, which is in contradiction to the definition of c'.

Assume for contradiction that the set C' of closed geodesics with $E(c') = \kappa'$ is empty. Then by Lemma 2 we have $\mathcal{U} = \emptyset$ and $\varepsilon > 0$ such that $\mathcal{D}(2, P^{\kappa'+\varepsilon}) \subset P^{\kappa'-\varepsilon}$. But since $\mathcal{D}(2, -)$ maps $P^{\kappa'}M$ into itself this is in contradiction with the minimality of κ'. Thus C' is non-empty and so there exists a closed geodesic.

Next assume that $\pi_1(M) = 0$. It's a fact from algebraic topology that on a compact manifold (without boundary) at least one homotopy group is non-zero. Thus choose smallest k such that $\pi_{k+1}(M) \neq 0$. Let $f : S^{k+1} \to M$ be a non-null homotopic differentiable map. This map induces a continuous map

$$F : D^k \to PM$$

by the following description: Identify D^k with the half-equator, $\{x \in S^{k+1} \mid x_0 \geq 0, x_1 = 1\}$, of S^{k+1} and associate to each $p \in D^k \subset S^{k+1}$ the parametrized circle $a_p(t)$, starting from p orthogonally to the hyperplane, $\{x \in S^{k+1} \mid x_1 = 0\}$, going into the half-sphere $\{x \in$

$S^{k+1} \mid x_1 \geq 0\}$. We then define $F(p) = f \circ a_p$. Note that $p \in \partial D^k$, then a_p is constant with $a_p(t) = p$.

We now consider a homotopy of F,

$$\Phi : [0,1] \times D^k \to PM$$

with $\Phi(0,-) = F$. This homotopy of F determines a homotopy, ϕ, of f (with $\phi(0,-) = f$) such that $F(\phi(t,-)) = \Phi(t,-)(f)$.

Indeed, since any $q \in S^{k+1}$ can be written as $q = a_p(t)$ for some $t \in S$ and $p \in D^k$, we may put

$$\phi(\sigma, q) = \phi(\sigma, a_p(t)) = \Phi(\sigma, p)(t),$$

where we use $\Phi(0,p) = F(p) = f \circ a_p = \phi(0,-) \circ a_p$, which holds not only for $\sigma = 0$ but any σ, i.e. $\Phi(\sigma, p) = \phi(\sigma, a_p)$. In particular, if F is null-homotopic, then f is also null-homotopic.

There exists a $\kappa > 0$ such that $E|_{F(D^k)} < \kappa$ and we choose $k \in 2\mathbb{N}$ such that $4\kappa/k \leq \eta^2$. Consider now the n-time repeated application of $\mathcal{D}(2,-)$ on $F(D^k)$ denoted by $\mathcal{D}^n(2, F(D^k))$, and the limit

$$\kappa_0 = \lim_{n\to\infty} \max E|_{\mathcal{D}^n(2,F(D^k))}$$

We want to show $\kappa_0 > 0$, so assume for contradiction that $\kappa_0 = 0$. As before we have that $\kappa_0 = 0$ implies that $L|_{\mathcal{D}^n(2,F(D^k))} \leq \sqrt{2E|_{\mathcal{D}^n(2,F(D^k))}} < \eta$. I.e. for any $p \in D^k$ we have that $\mathcal{D}^n(2, F(p))$ is a closed curve of length $< \eta$, which is, by choice of η, always contractable. Hence we have that F is homotopic to a map F^* with $F^*(D^k) \subset P^0M$, which implies that F is homotopic to a constant map with image a single point in P^0M (which can be identified with M). Thus also f must be homotopic to a constant map, which is in contradiction to the definition of f. So we must have $\kappa_0 > 0$.

Finally assume for contradiction that the set C' of closed geodesics wth $E(c') = \kappa_0$ is empty. Then by Lemma 2 we have $\mathcal{U} = 0$ and $\varepsilon > 0$ such that $\mathcal{D}(2, P^{\kappa_0+\varepsilon}M) \subset P^{\kappa_0-\varepsilon}M$, which is in contradiction to the definition of κ_0. Hence C' is non-empty and there exists a closed geodesic. □

In fact, in the case with $\pi_1(M) \neq 0$ any element of π_1 can be represented by a closed geodesic, as shown in Lee [2].

Note also that the condition that M has no boundary is nessecary for the proof since, for example, the disk, D^n, has $\pi_k(D^n) = 0$ for all $k > 0$.

One could continue expanding the Lyusternik-Schnirelmann theory to prove the theorem of Lyusternik and Schnirelmann (also know as the Theorem of Three Geodesics):

Theorem 3 (Theorem of Three Geodesics)**.** *On the 2-dimensional sphere with an arbitrary Riemannian metric, there exists three closed geodesics without self-intersections.*

(Note that a triaxial ellipsoid with all axes having approximately the same length will have no more than three such geodesics, meaning that in general no more than three such geodesics can be found).

However the proof of this fact is quite a bit more cumbersome, but can be found in [1].

References

1. Klingenberg, W. (1978). Lectures on closed geodesics. Grundlehren Der Mathematischen Wissenschaften. https://doi.org/10.1007/978-3-642-61881-9

2. Lee, J. M. (2018). Introduction to riemannian manifolds. Graduate Texts in Mathematics. https://doi.org/10.1007/978-3-319-91755-9